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Finding Cheeger Cuts via 1-Laplacian of Graphs

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Abstract. In this paper, we propose a novel algorithm for finding Cheeger cuts via 1-Laplacian of graphs. In [6], Chang introduced the theory of 1-Laplacian of graphs and built the connection between searching for the Cheeger cut of an undirected and unweighted graph and finding the first nonzero eigenvalue of 1-Laplacian, the latter of which is equivalent to solving a constrained non-convex optimization problem. We develop an alternating direction method of multipliers based algorithm to solve the optimization problem. We also prove that the generated sequence is bounded and it thus has a convergent subsequence. To find the goal optimal solution to the problem, we apply the proposed algorithm using different initial guesses and select the cut with the smallest cut value as the desired cut. Experimental results are presented for typical graphs, including Petersen's graph and Cockroach graphs, and the well-known Zachary karate club graph.

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Key words: Cheeger cuts, 1-Laplacian of graphs, alternating direction method of multipliers.

1. Introduction

Partitioning a large dataset into a prescribed number of subsets is a fundamental problem in machine learning and it has many applications in the fields ranging from computer sciences, statistics, computational biology, image processing, neural networks, and social sciences, etc. For instance, in the field of image processing, image segmentation can be regarded as a clustering problem that aims to partition a given image domain into several parts in order to capture target objects or extract features depicted in images. The community detection problem is also a clustering problem, aiming to divide a community into two or more smaller communities, each of which shares similar preference or homogeneity. For example, for the well-known Zachary's

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karate club network [25], there are 34 members and 78 edges, and each edge indicates the associated two members interacted or were friends. One interesting question is to split this network into two groups such that members in each group have some tight connection and members in different groups are only loosely connected. In chemical engineering, cutting 3D crystals into two separate partitions by severing a minimum number of bonds assumes a large variety of technological applications [23]. Those crystals can also be described as graphs with atoms for nodes and bonds for edges.

All the above mentioned datasets can be expressed as graphs with nodes and edges. The partition of those data sets is indeed a graph cut problem, whose objective is to split a data set into sensible subsets such that points or nodes in each subset share some similarity or homogeneity while points in different subsets are dissimilar. Spectral graph theory [10] is one of the most successful mathematical tools for tackling graph cut problems. It employs the characteristic polynomials, eigenvalues, and eigenvectors of matrices that are associated with a given graph, including the graph Laplacian matrix, adjacency matrix, and so on. Its appealing feature lies in the fact that the properties of a graph, such as connectivity and symmetry, can be determined by the spectrum of those matrices.

To find some appropriate cut, specific energy functions or cut functions are often designed for graphs. In the literature, many different cuts have been proposed, including the Cheeger cut [8], the ratio cut [13], the normalized cut [19], etc. The Cheeger cut is one of the most important cuts and it originates from the well-known Cheeger's inequality from Riemannian geometry, where Cheeger proved an inequality that involves the first nontrivial eigenvalue of the Laplace-Beltrami operator on a compact Riemannian manifold. The Cheeger cut is a discrete analogue that associates with the graph Laplacian matrix.

Solving those cut problems is usually NP-hard and one has to resort to approximate solutions. Spectral graph theory provides the most popular approaches for obtaining such approximate solutions of the original cut problems [10, 22]. By using spectral graph theory, the original cut problem is relaxed to some linear algebra problem that can be handled easily.

In the literature, lots of research works have focused on the development of relaxation of the cut problems [4, 5, 19, 20, 22], especially for the Cheeger cut. Buhler *et al.* [5] considered the spectral clustering based on the graph p-Laplacian with $p > 1$, and showed that the limit cut, as $p \to 1+$, of thresholding the second eigenvector of this p-Laplacian converges to the optimal Cheeger cut. In fact, Kawohl *et al.* [14, 15] proved that the Cheeger constant equals to the limit of the first eigenvalue of the p-Laplacian as $p \to 1+$. In a recent work by Bresson *et al.* [4], they proposed minimizing the original l^1-relaxation of the Cheeger cut by employing the augmented Lagrangian method [11]. Even though these relaxations have produced lots of promising clustering results for a variety of practical problems, they only provide approximations for Cheeger cuts. Recently, Chang [6] developed a novel nonlinear spectral graph theory and systematically studied the 1-Laplacian of graphs, including the associated eigenvalue problem and the structure of its solutions. Most importantly, in this work, for