

Analysis and Efficient Implementation of Quadratic Spline Collocation ADI Methods for Variable-Order Time-Fractional Mobile-Immobile Diffusion Equations

Jun Liu¹, Hongfei Fu^{2,3,*}, Bingyin Zhang² and Jiansong Zhang¹

¹ College of Science, China University of Petroleum (East China),
Qingdao 266580, China

² School of Mathematical Sciences, Ocean University of China,
Qingdao 266100, China

³ Laboratory of Marine Mathematics, Ocean University of China,
Qingdao 266100, China

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Abstract. In this paper, a quadratic spline collocation (QSC) method combined with L_1 time discretization in the framework of alternating direction implicit (ADI) approach, namely ADI-QSC- L_1 method, is developed to solve the variable-order time-fractional mobile-immobile diffusion equations in multi-dimensional spaces. Discrete L_2 norm-based stability and error estimate are carefully discussed, which show that the proposed method is unconditionally stable and convergent with first-order accuracy in time and second-order accuracy in space. Then, based on the exponential-sum-approximation technique for the fast evaluation of the variable-order Caputo fractional derivative, an efficient implementation strategy of the ADI-QSC- L_1 method, named ADI-QSC- FL_1 is presented, which further improves the computational efficiency by reduced memory requirement and computational cost. Finally, numerical examples are provided to support both the theoretical results and efficiency of the developed method.

AMS subject classifications: 65M12, 65M15, 65M70

Key words: Time-fractional mobile-immobile diffusion equations, variable-order, QSC method, ADI, stability and convergence, fast implementation.

1. Introduction

A large number of research indicates that many natural phenomena and structures can be better described by fractional differential equations (FDEs), due to the histori-

*Corresponding author. Email addresses: liujun@upc.edu.cn (J. Liu), fhf@ouc.edu.cn (H. Fu), zhangbingyin@stu.ouc.edu.cn (B. Zhang), jszhang@upc.edu.cn (J. Zhang)

cal memory and global correlation of the fractional operators, such as fluid flow in an unsaturated media [32, 36], viscoelastic anomalous diffusion in complex liquids [30], chemical reactions in underground water and so on [37]. In recent years, the time-FDEs, which are often used to describe the subdiffusive solutes transport in heterogeneous media [31, 34, 46] or model the memory behavior of shape-memory polymer [17], have attracted great interests of researchers. Wang and Zheng [40] developed a modified two-scale variable-order time-fractional mobile-immobile equation to model the solute transport in heterogeneous porous media, and they rigorously proved the wellposedness and regularity of the model.

It is well-known that numerical modeling is one of the main methods for solving many types of FDEs, and indeed various numerical methods have been developed, such as finite difference methods [1, 9, 14, 20, 28, 35, 38, 39, 51], finite element methods [3, 13, 15, 26, 43, 48], finite volume methods [7, 8, 49], spectral methods [2, 44]. As an efficient numerical tool, quadratic spline collocation (QSC) methods have been successfully studied for integer-order differential equations [4, 5, 10], and have also been applied for FDEs, for example, Luo *et al.* [27] proposed a space-time QSC method in all-at-once manner to solve the sub-diffusion equations. We combined the QSC method and $L1$ formula for solving the sub-diffusion equation with fractional boundary conditions, and rigorous numerical analysis is given in [24]. Besides, we also developed and analyzed the QSC method for space-FDEs [22, 23, 25]. Moreover, some other types of collocation methods for FDEs are also considered, see [16, 18, 42]. Recently, the authors applied the QSC method for variable-order time-fractional mobile-immobile diffusion equations with variably diffusive coefficients [21], and showed that the method is unconditionally stable and convergent with first-order in time and second-order in space with respect to some discrete and continuous L_2 norms. Then, combined with the reduced basis technique, an efficient QSC-L1-RB method was proposed to further improve the computational efficiency. This seems the first paper on analysis of QSC method for variable-order time-fractional model. However, for large-scale and long time modeling and simulations, the method is still computationally expensive. Therefore, it deserves to develop much more efficient numerical methods for the interested two-scale variable-order model.

Alternating direction implicit (ADI) method, serving an operator splitting method, is able to convert the multi-dimensional large-scale problem into a series of one-dimensional small-scale subproblems. Thus, it can reduce the computational cost greatly and solve the model problem easily in parallel. Various ADI methods have been extensively developed for time-FDEs [6, 41, 47]. Another efficient approach for time-FDEs is the sum-of-exponentials (SOE) technique. Jiang *et al.* [12] proposed a fast evaluation method for the constant-order Caputo fractional derivative by the SOE approximation. This method is able to reduce the memory requirement and the computational cost greatly. However, it can not be directly applied to the variable-order fractional derivative and related model. Recently, Zhang *et al.* [45] provided a variant SOE technique, named exponential-sum-approximation (ESA), to approximate the variable-order Caputo fractional derivative and variable-order time-fractional diffusion equation. In this