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Two Dynamical Models Based on Projection Operator for Solving the System of Absolute Value Equations Associated with Second-Order Cone

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Abstract. A new equivalent reformulation of the absolute value equations associated with second-order cone (SOCAVEs) is emphasised, from which two dynamical models based on projection operator for solving SOCAVEs are constructed. Under suitable conditions, it is proved that the equilibrium points of the dynamical systems exist and could be (globally) asymptotically stable. The effectiveness of the proposed methods are illustrated by some numerical simulations.

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1. Introduction

The second-order cone (SOC) in $\mathbb{R}^n$ is defined by

$$\mathcal{K}^n = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x_2\| \le x_1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, let $\mathcal{K}^n$ represent the set of nonnegative reals. Moreover, a general SOC $\mathcal{K} \subset \mathbb{R}^n$ could be the Cartesian product of some SOCs [10, 11, 15], i.e.

$$\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_r}$$

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with $n_1, \cdots, n_r, r \geq 1$ and $n_1 + \cdots + n_r = n$. Without loss of generality, we focus on the case that $r = 1$ because all the analysis can be carried over to the setting of $r > 1$ according to the property of the Cartesian product. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, their Jordan product is defined as [10, 11, 15]

$$x \circ y = (\langle x, y \rangle, y_1 x_2 + x_1 y_2) \in \mathbb{R} \times \mathbb{R}^{n-1},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. With this definition, the absolute value vector $|x|$ in SOC $\mathcal{K}^n$ is computed by

$$|x| = \sqrt{x \circ x}. \tag{1.1}$$

In this paper, we consider the problem of solving the absolute value equations associated with SOC (SOCAVEs) of the form

$$Ax - |x| - b = 0 \tag{1.2}$$

with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Unless otherwise stated, throughout this paper, $|x|$ is defined as in (1.1). SOCAVEs (1.2) is a special case of the generalized absolute value equations associated with SOC (SOCGAVEs)

$$Cx + D|x| - c = 0 \tag{1.3}$$

with $C, D \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. To the best of our knowledge, SOCGAVEs (1.3) was formally introduced by Hu *et al.* [20] and further studied in [23, 39, 40, 42] and the references therein. In addition, SOCAVEs (1.2) is a natural extension of the standard absolute value equations (AVEs)

$$Ax - |x| = b, \tag{1.4}$$

meanwhile, SOCGAVEs (1.3) is an extension of the generalized absolute value equations (GAVEs)

$$Cx + D|x| = c. \tag{1.5}$$

In AVEs (1.4) and GAVEs (1.5), the vector $|x|$ denotes the componentwise absolute value of the vector $x \in \mathbb{R}^n$. It is known that GAVEs (1.5) with $m = n$ was first introduced by Rohn in [44] and further investigated in [18, 31, 43] and the references therein. Obviously, AVEs (1.4) is a special case of GAVEs (1.5).

Over the past two decades, AVEs (1.4) and GAVEs (1.5) have been widely studied because of their relevance to many mathematical programming problems, such as the linear complementarity problem (LCP), the bimatrix game and others, see e.g. [31, 34, 43]. Hence, abundant theoretical results and numerical algorithms for both AVEs (1.4) and GAVEs (1.5) have been established. On the theoretical aspect, for instance, Mangasarian [31] showed that solving GAVEs (1.5) is NP-hard; if GAVEs (1.5) has a solution, checking whether it has a unique solution or multiple solutions is NP-complete [43]. Moreover, various sufficient or necessary conditions on solvability and non-solvability of AVEs (1.4) and GAVEs (1.5) were discussed in [19,34,37,43,45]. The latest trend is to investigate the error bound and the condition number of AVEs (1.4) [51]. On the numerical aspect, there are many algorithms for solving AVEs (1.4) and GAVEs (1.5). For instance, the Newton-type methods [3, 4, 33], the SOR-like method [6, 25], the concave minimization methods [32, 50], the exact and inexact Douglas-Rachford splitting methods [8] and others, see e.g. [1, 7, 29, 35, 36, 47–49] and the references therein.

We are interested in SOCAVEs (1.2) and SOCGAVEs (1.3) not only because they are extensions of the standard ones, but also because they are equivalent with some LCPs associated with SOC (SOCLCPs), which have various applications in engineering, control and finance [20,38,40]. Recently, some numerical methods and theoretical results have been developed for SOCAVEs (1.2) and SOCGAVEs (1.3). For the numerical side, Hu *et al.* [20] proposed a generalized Newton method for solving SOCGAVEs (1.3) (here and in the sequel, we assume $m = n$). Then, Huang and Ma [23] presented some weaker convergent conditions of the generalized Newton method. Miao *et al.* [40] proposed a smoothing Newton method for SOCGAVEs (1.3) and a unified way to construct smoothing functions is explored in [42]. Huang and Li [22] proposed a modified SOR-like method for SOCAVEs (1.2). Miao *et al.* [41] suggested a Levenberg-Marquardt method with Armijo line search for SOCAVEs (1.2). For the theoretical side, Miao *et al.* [39] studied the existence and nonexistence of solution to SOCAVEs (1.2) and SOCGAVEs (1.3). In addition, the unique solvability for SOCAVEs (1.2) and SOCGAVEs (1.3) was also investigated in [39]. Miao and Chen [38] investigated conditions under which the unique solution of SOCAVEs (1.2) is guaranteed, which are different from those in [39]. Hu *et al.* proved that SOCGAVEs (1.3) is equivalent to the following problem: Find $x, y \in \mathbb{R}^n$ such that

$$\begin{aligned} &Mx + Py = p, \\ &x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n, \quad \langle x, y \rangle = 0, \end{aligned} \tag{1.6}$$

where $M, P \in \mathbb{R}^{n\times n}$ and $p \in \mathbb{R}^n$. However, the problem (1.6) is not a standard SOCLCP, which is in the form of

$$z \in \mathcal{K}^\ell, \quad w = Nz + q \in \mathcal{K}^\ell, \quad \langle z, w \rangle = 0, \tag{1.7}$$

where $N \in \mathbb{R}^{\ell\times\ell}$ and $q \in \mathbb{R}^\ell$. Miao *et al.* [40] showed that SOCGAVEs (1.3) is equivalent to SOCLCP (1.7) with

$$N = \begin{bmatrix} -I & 2I & 0 \\ C & D-C & 0 \\ -C & C-D & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 2x_+ \\ |x| \\ 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -c \\ c \end{bmatrix},$$

where x_+ is the projection of x onto the SOC $\mathcal{K}^n$. Note that the above matrix N has three times the dimension of the matrix A (or B) (i.e., $\ell = 3n$). More recently, Miao and Chen [38], under the condition that 1 is not an eigenvalue of A or N, provided the equivalence between SOCAVEs (1.2) and SOCLCP (1.7) without changing the dimension (i.e., $\ell = n$). The goals of this paper are twofold: to highlight another equivalent reformulation of SOCAVEs (1.2) and to present two dynamical models to solve SOCAVEs (1.2). In contrast to the numerical methods mentioned above, our methods are from a continuous perspective. Our work here is inspired by recent studies on AVEs (1.4) [7].

The rest of this paper is organized as follows. In Section 2, a few relevant basic results on the SOC and the autonomous system are introduced. In Section 3, an equivalent reformulation of SOCAVEs (1.2) is recalled, and two dynamical models and their stability analysis are given. Numerical simulations are given in Section 4. Conclusions are made in Section 5.

Notation. The set of all $n \times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. We use I to denote the identity matrix with suitable dimension. The transposition of a matrix or a vector is denoted by $\cdot^\top$. The inner product of two vectors in $\mathbb{R}^n$ is defined as

$$\langle x, y\rangle \doteq \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad \|x\| \doteq \sqrt{\langle x, x\rangle}.$$

The spectral norm of A is denoted by $\|A\|$ and is defined by the formula

$$\|A\| \doteq \max\big\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\big\}.$$

We use $\mathrm{tridiag}(a, b, c)$ to denote a tridiagonal matrix, which has a, b, c as the subdiagonal, main diagonal and superdiagonal entries, respectively. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite if $\langle Ax, x\rangle > 0$ for all $0 \neq x \in \mathbb{R}^n$.

2. Preliminaries

In this section, we collect some results which lay the foundation of our later analysis. We first recall some basic concepts and background materials regarding SOCs, which can be found in [2, 10, 11, 14, 15].

For $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the spectral decomposition of x with respect to SOC is given by

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \tag{2.1}$$

where

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|,$$
$$u_x^{(i)} = \begin{cases} \dfrac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\|x_2\|}\right), & \text{if} \quad x_2 \neq 0, \\ \dfrac{1}{2}\left(1, (-1)^i w\right), & \text{if} \quad x_2 = 0 \end{cases}$$

for $i = 1, 2$ and w is any vector in $\mathbb{R}^{n-1}$ with $\|w\| = 1$. If $x_2 \neq 0$, the spectral decomposition is unique. We call $\lambda_1(x)$ and $\lambda_2(x)$ the eigenvalues of x and $\{u_x^{(1)}, u_x^{(2)}\}$ is called a Jordan frame of x. It is known that $\lambda_1(x)$ and $\lambda_2(x)$ are nonnegative if and only if $x \in \mathcal{K}^n$. For any real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define a function on $\mathbb{R}^n$ associated with $\mathcal{K}^n$ by

$$f(x) \doteq f\big(\lambda_1(x)\big)u_x^{(1)} + f\big(\lambda_2(x)\big)u_x^{(2)},$$

if $x \in \mathbb{R}^n$ has the spectral decomposition (2.1). Then we have

$$|x| = \begin{cases} \dfrac{1}{2}\Bigg(|x_1 - \|x_2\|| + |x_1 + \|x_2\||, \\ \qquad (|x_1 + \|x_2\|| - |x_1 - \|x_2\||)\, \dfrac{x_2}{\|x_2\|}\Bigg), & \text{if} \quad x_2 \neq 0, \\ (|x_1|, 0), & \text{if} \quad x_2 = 0. \end{cases} \tag{2.2}$$

The projection mapping from $\mathbb{R}^n$ onto $\Omega \subset \mathbb{R}^n$, denoted by P_Ω, is defined as

$$P_\Omega(x) \doteq \arg\min\big\{\|x - y\| : y \in \Omega\big\}.$$

Given $u \in \mathbb{R}^n$ and a nonempty closed convex subset Ω of $\mathbb{R}^n$, μ is the projection of u onto Ω, i.e., $\mu = P_\Omega(u)$ if and only if (see e.g. [5, Theorem 1.2.4])

$$\langle u - \mu, v - \mu \rangle \leq 0, \quad \forall v \in \Omega. \tag{2.3}$$

As mentioned earlier, let x_+ be the projection of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ onto $\mathcal{K}^n$, then we have

$$x_+ = \begin{cases} x, & \text{if} \quad x \in \mathcal{K}^n, \\ 0, & \text{if} \quad x \in -\mathcal{K}^n, \\ u, & \text{otherwise}, \end{cases} \tag{2.4}$$

where

$$u = \begin{bmatrix} \dfrac{x_1 + \|x_2\|}{2} \\ \left(\dfrac{x_1 + \|x_2\|}{2}\right) \dfrac{x_2}{\|x_2\|} \end{bmatrix}.$$

The dual cone of $\mathcal{K}^n$ is defined as

$$(\mathcal{K}^n)^* \doteq \big\{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in \mathcal{K}^n\big\}.$$

It is known that SOC $\mathcal{K}^n$ is a pointed close convex cone and it is self-dual (i.e., $(\mathcal{K}^n)^* = \mathcal{K}^n$).

Now we turn to the autonomous system. Consider the autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), \tag{2.5}$$

where g is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Throughout this paper, we use $x(t; x(t_0))$ to denote the solution of (2.5) determined by the initial value condition $x(t_0) = x_0$. The following results are well-known and can be found in [27, Chapters 2,3].

Definition 2.1. *The function $F:\mathbb{R}^n\to\mathbb{R}^n$ is said to be Lipschitz continuous with Lipschitz constant $L>0$ if*

$$\|F(x)-F(y)\|\le L\|x-y\|,\quad \forall x,y\in\mathbb{R}^n.$$

Lemma 2.1. *Assume that $g:\mathbb{R}^n\to\mathbb{R}^n$ is Lipschitz continuous in $\mathbb{R}^n$, then for arbitrary $t_0\ge 0$ and $x(t_0)=x_0\in\mathbb{R}^n$, the dynamical system* (2.5) *has a unique solution $x(t;x(t_0))$, $t\in[t_0,+\infty)$.*

Definition 2.2 (Equilibrium Point)**.** *A vector $x^*\in\mathbb{R}^n$ is called an equilibrium point of the dynamical system* (2.5) *if $g(x^*)=0$.*

Definition 2.3. *The equilibrium point x^* of* (2.5) *is stable if, for any $\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that*

$$\|x(t_0)-x^*\|<\delta\quad\Rightarrow\quad\|x(t;x(t_0))-x^*\|<\epsilon,\quad\forall t\ge t_0.$$

Furthermore, the equilibrium point x^* of (2.5) *is asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(t_0)-x^*\|<\delta\quad\Rightarrow\quad\lim_{t\to\infty}x\big(t;x(t_0)\big)=x^*.$$

Theorem 2.1. *Let x^* be an equilibrium point of* (2.5) *and $\Omega\subseteq\mathbb{R}^n$ be a domain containing x^*. If there is a continuously differentiable function $V:\Omega\to\mathbb{R}$ such that*

$$\begin{aligned}&V(x^*)=0,\quad V(x)>0, &&\forall x\in\Omega\backslash\{x^*\},\\&\frac{\mathrm{d}V(x)}{\mathrm{d}t}=\nabla V(x)^\top g(x)\le 0, &&\forall x\in\Omega,\end{aligned}$$

then x^* is stable. Moreover, if

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t}<0,\quad\forall x\in\Omega\backslash\{x^*\},$$

then x^* is asymptotically stable.

Theorem 2.2. *Let x^* be an equilibrium point for* (2.5). *If there exists a continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}$ such that*

$$\begin{aligned}&V(x^*)=0,\quad V(x)>0,\quad \forall x\ne x^*,\\&\frac{\mathrm{d}V(x)}{\mathrm{d}t}<0,\qquad\qquad\quad\ \forall x\ne x^*,\\&\|x-x^*\|\to\infty\quad\Rightarrow\quad V(x)\to\infty,\end{aligned}$$

then x^* is globally asymptotically stable.

3. The equivalent reformulation and the dynamical models

In this section, we first highlight that SOCAVEs (1.2) is equivalent to the generalized SOCLCP (SOCGLCP) as follows:

$$S(x) \doteq Ax + x - b \in \mathcal{K}^n, \quad T(x) \doteq Ax - x - b \in \mathcal{K}^n, \quad \langle S(x), T(x) \rangle = 0. \tag{3.1}$$

Then, two novel dynamical models are presented to solve SOCAVEs (1.2).

In order to claim the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1), we introduce the following two lemmas.

Lemma 3.1 ([13, 31]). *Let $a, b \in \mathbb{R}$. Then $a \geq 0, b \geq 0$ and $ab = 0$ if and only if $a + b = |a - b|$.*

Lemma 3.2. *Let $s, t \in \mathbb{R}^n$. Then*

$$\langle s, t \rangle = 0, \quad s \in \mathcal{K}^n, \quad t \in \mathcal{K}^n, \tag{3.2}$$

if and only if

$$s + t = |s - t|. \tag{3.3}$$

Proof. We first prove that (3.2) $\Rightarrow$ (3.3). Since $s = (s_1, s_2) \in \mathcal{K}^n$ and $t = (t_1, t_2) \in \mathcal{K}^n$, we have $s_1 \geq \|s_2\|$ and $t_1 \geq \|t_2\|$, which implies that

$$|\langle s_2, t_2 \rangle| \leq \|s_2\| \|t_2\| \leq s_1 t_1.$$

Thus,

$$\langle s, t \rangle = s_1 t_1 + s_2^\top t_2 \geq s_1 t_1 - s_1 t_1 = 0,$$

and the equality is valid if and only if $s_2 = k t_2$ $(k \geq 0)$, $s_1 = \|s_2\|$ and $t_1 = \|t_2\|$. Hence, the vectors s and t in (3.2) share the same Jordan frame [2, 38]. Let $s = \lambda_1 e_1 + \lambda_2 e_2$ and $t = \mu_1 e_1 + \mu_2 e_2$, where $\{e_1, e_2\}$ is the Jordan frame. Then we have $\lambda_i, \mu_i \geq 0$ for $i = 1, 2$ and $\lambda_1 \mu_1 = \lambda_2 \mu_2 = 0$. It then follows from Lemma 3.1 that

$$\lambda_1 + \mu_1 = |\lambda_1 - \mu_1|, \quad \lambda_2 + \mu_2 = |\lambda_2 - \mu_2|.$$

On the other hand, we have

$$\begin{aligned} s + t &= (\lambda_1 + \mu_1) e_1 + (\lambda_2 + \mu_2) e_2, \\ |s - t| &= |\lambda_1 - \mu_1| e_1 + |\lambda_2 - \mu_2| e_2. \end{aligned}$$

Hence, we have (3.3).

Next, we prove that (3.3) $\Rightarrow$ (3.2). By (3.3), we know that $s + t$ and $s - t$ have the same Jordan frame, from which we obtain that s and t have the same Jordan frame. Indeed, it follows from the fact that

$$2s = (s + t) + (s - t), \quad 2t = (s + t) - (s - t).$$

Let $s = \lambda_1 e_1 + \lambda_2 e_2$ and $t = \mu_1 e_1 + \mu_2 e_2$. Then it follows from (3.3) that $\lambda_1 + \mu_1 = |\lambda_1 - \mu_1|$ and $\lambda_2 + \mu_2 = |\lambda_2 - \mu_2|$, which combine with Lemma 3.1 implies that $\lambda_i, \mu_i \geq 0$ for $i = 1, 2$ and $\lambda_1 \mu_1 = \lambda_2 \mu_2 = 0$. Then we can obtain (3.2). $\square$

Remark 3.1. Lemma 3.2 can be found in [15, Proposition 4.1] and [12, Proposition 2.2(d)]. Here we give a new proof based on the Jordan frame, which is different from that of [15, Proposition 4.1].

According to Lemma 3.2, if we set $s+t=Ax-b$ and $s-t=x$, we can obtain the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1). We should point out that the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1) is implicit in the proof of [38, Theorem 4.1] and the proof of Lemma 3.2 is also inspired by that of [38, Theorem 4.1]. Moreover, SOCGLCP (3.1) is equivalent to the generalized linear variational inequality problem associated with SOC (SOCGLVI) [13]: Find $x^* \in \mathbb{R}^n$ such that

$$\langle v - S(x^*), T(x^*)\rangle \geq 0, \quad S(x^*) \in \mathcal{K}^n, \quad \forall v \in \mathcal{K}^n, \tag{3.4}$$

which is also equivalent to finding a solution of

$$S(x) = P_{\mathcal{K}^n}\left[S(x) - T(x)\right]. \tag{3.5}$$

Indeed, we have the following theorem.

Theorem 3.1. *The vector* x^* *solves SOCAVEs* (1.2) *if and only if* $r(x^*)=0$*, where*

$$r(x) = S(x) - P_{\mathcal{K}^n}\left[S(x) - T(x)\right]. \tag{3.6}$$

Furthermore, it can be proved that

$$r(x) = Ax - |x| - b. \tag{3.7}$$

Proof. It is enough to prove (3.7). Note that $S(x)-T(x)=2x$. We will split the proof into three cases.

(a) If $x \in \mathcal{K}^n$, then $2x \in \mathcal{K}^n$. It follows from $|x|=x$ and $P_{\mathcal{K}^n}(2x)=2x$ that

$$S(x) - P_{\mathcal{K}^n}\left[S(x)-T(x)\right] = Ax + x - b - 2x = Ax - x - b = Ax - |x| - b.$$

(b) If $x \in -\mathcal{K}^n$, $2x \in -\mathcal{K}^n$. Then it follows from $|x|=-x$ and $P_{\mathcal{K}^n}(2x)=0$ that

$$S(x) - P_{\mathcal{K}^n}\left[S(x)-T(x)\right] = Ax + x - b - 0 = Ax - (-x) - b = Ax - |x| - b.$$

(c) If $x \notin \mathcal{K}^n$ and $x \notin -\mathcal{K}^n$, it follows from (2.4) that

$$P_{\mathcal{K}^n}(2x) = \begin{bmatrix} x_1 + \|x_2\| \\ \dfrac{x_1 x_2}{\|x_2\|} + x_2 \end{bmatrix}.$$

In addition, it follows from (2.2) that

$$|x| = \begin{bmatrix} \|x_2\| \\ \dfrac{x_1 x_2}{\|x_2\|} \end{bmatrix}.$$

Then

$$\begin{aligned}
&S(x) - P_{\mathcal{K}^n}\left[S(x) - T(x)\right] \\
&= Ax + x - b - \begin{bmatrix} x_1 + \|x_2\| \\ \dfrac{x_1 x_2}{\|x_2\|} + x_2 \end{bmatrix} \\
&= Ax - \begin{bmatrix} \|x_2\| \\ \dfrac{x_1 x_2}{\|x_2\|} \end{bmatrix} - b = Ax - |x| - b.
\end{aligned}$$

The proof is complete. □

Remark 3.2. An anonymous referee points out a simpler way to prove Theorem 3.1, which is described as follows. Since [9, Property 1.2(f)]

$$P_{\mathcal{K}^n}(x) = \frac{1}{2}(x + |x|),$$

it holds that

$$r(x) = S(x) - P_{\mathcal{K}^n}\left[S(x) - T(x)\right] = Ax + x - b - \frac{1}{2}(2x + |2x|) = Ax - |x| - b.$$

In order to consider the stability of the equilibrium points of the dynamical systems proposed below, we need the following theorem.

Theorem 3.2. *If x^* is a solution of SOCAVEs* (1.2) *and* $\|A^{-1}\| \leq 1$, *then*

$$(x - x^*)^\top A^\top r(x) \geq \frac{1}{2}\left\|r(x)\right\|^2, \quad \forall x \in \mathbb{R}^n. \tag{3.8}$$

Proof. The proof is inspired by that of [17, Theorem 2]. Since $\mathcal{K}^n$ is a closed convex set and $S(x^*) \in \mathcal{K}^n$, it follows from (2.3) that

$$\left[v - P_{\mathcal{K}^n}(v)\right]^\top \left[P_{\mathcal{K}^n}(v) - S(x^*)\right] \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Let $v \doteq S(x) - T(x)$, we have

$$\left[r(x) - T(x)\right]^\top \{P_{\mathcal{K}^n}[S(x) - T(x)] - S(x^*)\} \geq 0. \tag{3.9}$$

On the other hand, it follows from

$$P_{\mathcal{K}^n}(\cdot) \in \mathcal{K}^n, \quad T(x^*) \in \mathcal{K}^n = (\mathcal{K}^n)^*, \quad S(x^*)^\top T(x^*) = 0,$$

that

$$T(x^*)^\top \{P_{\mathcal{K}^n}[S(x) - T(x)] - S(x^*)\} \geq 0. \tag{3.10}$$

It follows from (3.9), (3.10) and

$$P_{\mathcal{K}^n}[S(x) - T(x)] - S(x^*) = [S(x) - S(x^*)] - r(x)$$

that

$$\begin{aligned}&\left\{[S(x) - S(x^*)] + [T(x) - T(x^*)]\right\}^\top r(x)\\ \geq\ &\|r(x)\|^2 + [S(x) - S(x^*)]^\top [T(x) - T(x^*)],\end{aligned}$$

which together with the definitions of S and T in (3.1) implies

$$2r(x)^\top A(x - x^*) \geq \|r(x)\|^2 + (x - x^*)^\top (A^\top A - I)(x - x^*), \quad \forall x \in \mathbb{R}^n.$$

Then the proof is completed with $\|A^{-1}\| \leq 1$. □

3.1. The first dynamical model for SOCAVEs

Now we are in the position to develop a dynamical system to solve SOCAVEs (1.2). Inspired by [16,21,30,46], we propose the following projection-type dynamical system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma A^\top \left\{P_{\mathcal{K}^n}\big[S(x) - T(x)\big] - S(x)\right\}, \tag{3.11}$$

where $\gamma > 0$ is a constant. According to (3.6) and (3.7), the dynamical system (3.11) can be reduced to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma A^\top (b + |x| - Ax) \doteq h(x). \tag{3.12}$$

Based on Theorem 3.1, we have the following theorem.

Theorem 3.3. *Let A be nonsingular, then x^* is a solution of SOCAVEs* (1.2) *if and only if x^* is an equilibrium point of the dynamical system* (3.12).

Before ending this subsection, we will study the existence of the solutions and the stability of the equilibrium points of the dynamical system (3.12).

Lemma 3.3. *The function h defined as in* (3.12) *is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant $\gamma\|A^\top\|(\|A\| + 1)$.*

Proof. Since $\||x_1| - |x_2|\| \leq \|x_1 - x_2\|$ [23, 26, 39], the proof is trivial according to [7, Lemma 3.1]. □

Based on Lemmas 2.1 and 3.3, we have the following theorem.

Theorem 3.4. *For a given initial value $x(t_0) = x_0$, there exists a unique solution $x(t; x(t_0))$, $t \in [t_0, +\infty)$ for the dynamical system* (3.12).

Now we can give the following stability theorem.

Theorem 3.5. *Let $\|A^{-1}\| \leq 1$, then the equilibrium point x^* (if it exists) of the dynamical system* (3.12) *is asymptotically stable. In particular, if $\|A^{-1}\| < 1$, then the unique equilibrium point x^* of the dynamical system* (3.12) *is globally asymptotically stable.*

Proof. Let $x = x(t; x(t_0))$ be the solution of (3.12) with initial value $x(t_0) = x_0$ and x^* is the equilibrium point nearby x_0. Let

$$V(x) = \frac{1}{2}\|x - x^*\|^2, \quad x \in \mathbb{R}^n.$$

It is easy to check that $V(x^*) = 0$ and $V(x) > 0$ for all $x \neq x^*$. Moreover, it follows from (3.8) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x) &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -\gamma(x - x^*)^\top A^\top r(x) \\ &\leq -\frac{\gamma}{2}\|r(x)\|^2 < 0, \quad \forall x \neq x^*.\end{aligned}$$

Hence, the first part of the theorem follows from Theorem 2.1.

If $\|A^{-1}\| < 1$, SOCAVEs (1.2) has a unique solution [39] and thus the equilibrium point of (3.12) is unique. Since $V(x) \to \infty$ as $\|x - x^*\| \to \infty$, it follows from Theorem 2.2 that the unique equilibrium point is globally asymptotically stable. ☐

Remark 3.3. If $\|A^{-1}\| = 1$, then SOCAVEs (1.2) may have no solutions, more than one solutions or a unique solution (see Section 4 for more detail).

3.2. The second dynamical model for SOCAVEs

In this subsection, the dynamical model (3.12) is simplified in order to reduce computation. As a trade-off, we require the coefficient matrix A to be symmetric and positive definite. Concretely, based on Theorem 3.1, we propose another projection-type dynamical system to solve SOCAVEs (1.2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma\left\{P_{\mathcal{K}^n}\left[S(x) - T(x)\right] - S(x)\right\}, \tag{3.13}$$

where $\gamma > 0$ is a constant. Substituting (3.6) and (3.7) into (3.13), we obtain†

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma(b + |x| - Ax) \doteq l(x). \tag{3.14}$$

It follows from Theorem 3.1 that the equilibrium point of (3.14) equals the solution to SOCAVEs (1.2). That is, we have the following theorem.

†Comparing (3.12) and (3.14), it is easy to find that the structure of (3.14) is simpler than that of (3.12), which frequently results in (3.14) having a lower computational cost than (3.12).

Theorem 3.6. *x^* is a solution of SOCAVEs* (1.2) *if and only if x^* is an equilibrium point of the dynamical system* (3.14).

In the following, we will study the existence of the solution and the stability of the equilibrium point of the dynamical system (3.14).

Similar to the proof of Lemma 3.3, we have the following lemma.

Lemma 3.4. *The function l defined as in* (3.14) *is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant $\gamma(\|A\|+1)$.*

Based on Lemmas 2.1 and 3.4, the following theorem can be concluded.

Theorem 3.7. *For a given initial value $x(t_0)=x_0$, there exists a unique solution $x(t;x(t_0))$, $t\in[t_0,+\infty)$ for the dynamical system* (3.14).

In the following, we can give the stability theorem of the dynamical system (3.14).

Theorem 3.8. *If the coefficient matrix A is symmetric and positive definite and $\|A^{-1}\|\leq 1$, then the equilibrium point x^* (if it exists) of the dynamical system* (3.14) *is asymptotically stable. In particular, if A is symmetric and positive definite and $\|A^{-1}\|<1$, then the unique equilibrium point x^* of the dynamical system* (3.14) *is globally asymptotically stable.*

Proof. Let $x=x(t;x(t_0))$ be the solution of (3.14) with initial value $x(t_0)=x_0$ and x^* is the equilibrium point nearby x_0. Consider the following Lyapunov function:

$$\tilde{V}(x)=e^{(x-x^*)^\top A(x-x^*)}-1,\quad x\in\mathbb{R}^n.$$

It is easy to see that $\tilde{V}(x^*)=0$ and A is positive definite implies that $\tilde{V}(x)>0$ for all $x\neq x^*$. Moreover, it follows from (3.8), (3.14) and the symmetry of A that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}(x)&=\frac{\mathrm{d}\tilde{V}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\&=-\gamma e^{(x-x^*)^\top A(x-x^*)}\big[A(x-x^*)+A^\top(x-x^*)\big]^\top r(x)\\&\leq-\gamma e^{(x-x^*)^\top A(x-x^*)}\|r(x)\|^2<0,\quad\forall x\neq x^*.\end{aligned}$$

Hence, the first part of the theorem follows from Theorem 2.1. The remainder part of the proof is similar to that of Theorem 3.5. ☐

4. Numerical simulations

In this section, we will present four examples to illustrate the effectiveness of the proposed methods. All experiments are implemented in MATLAB R2018b with a machine precision 2.22×10^{-16} on a PC Windows 10 operating system with an Intel i7-9700 CPU and 8GB RAM. The involved systems of ordinary differential equations can be solved by the MATLAB "ode23" routine, which uses the embedded Fehlberg $(2,3)$ pair of explicit Runge-Kutta methods in extrapolated error-per-step (XEPS) mode [28]. More concretely, we use the MATLAB built-in expression

$$[t, y] = \text{ode23}(odefun, tspan, y_0),$$

which integrates the system of differential equations from t_0 to t_f with $tspan = [t_0, t_f]$. In general, it is not an easy task to choose t_f efficiently since the upper bound of the settling time is not a prior for our models. In the following, we select t_f by the trial-and-error method until the solution satisfies the termination criterion.

Example 4.1 ([23]). Consider SOCAVEs (1.2) with

$$A = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{n\times n}, \quad b = Ax^* - |x^*|,$$

where

$$x^* = (-1, 1, -1, 1, \cdots, -1, 1)^\top \in \mathbb{R}^n.$$

In this example, $\|A^{-1}\| < 1$ and A is symmetric and positive definite. Thus, SOCAVEs (1.2) has a unique solution for any $b \in \mathbb{R}^n$. Equivalently, both (3.12) and (3.14) have a unique equilibrium point and its globally asymptotical stability will be numerically checked. We first set $t_0 = 0$ and $t_f = 0.3$ for this example. In Fig. 1, we show the phase diagram of the state $x(t)$ with different initial points for $n = 2$ and $n = 3$, which visually display the globally asymptotical stability of the unique equilibrium point of (3.12) and (3.14). In Fig. 2, we show the influence of the parameter γ for (3.12) and (3.14), from which we find that the larger γ is, the faster convergence is. The same phenomenon occurred in some existing works, such as [7, 24].

In the following, we compare the efficiency of (3.12) and (3.14) by running them such that

$$\text{Res} = \frac{\|Ax - |x| - b\|}{\|b\|} \le 10^{-6}.$$

Here, we set $t_0 = 0$, $\gamma = 2$ and $x_0 = (0, 0, \cdots, 0)^\top$ for both models. Numerical results are reported in Table 1, from which we can conclude that (3.14) is better than (3.12) in terms of CPU (the elapsed CPU time in seconds), especially for large scale problems. This means that we can benefit from exploiting the inherent structure of the problem.

In the following, we will give three examples for $\|A^{-1}\| = 1$, in which SOCAVEs (1.2) may have more than one solutions, a unique solution or no solutions. In the following examples, $\gamma = 2$ is used.

Example 4.2. Consider SOCAVEs (1.2) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Obviously, SOCAVEs (1.2) has infinitely many solutions for this example. Thus, there are infinitely many equilibrium points to (3.12) and (3.14), respectively. In fact,

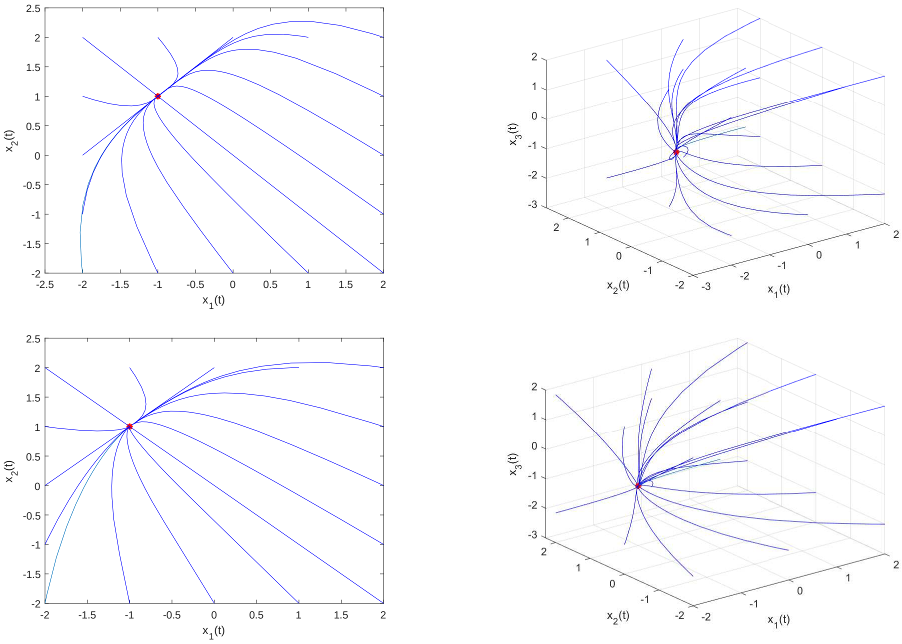

Figure 1: Phase diagrams of (3.12) (the up two subfigures) and (3.14) (the below two subfigures) for Example 4.1 (Left figures: $n = 2$ and 16 different initial points are used; Right figures: $n = 3$ and 20 different initial points are used). The exact equilibrium point is marked as red star point.

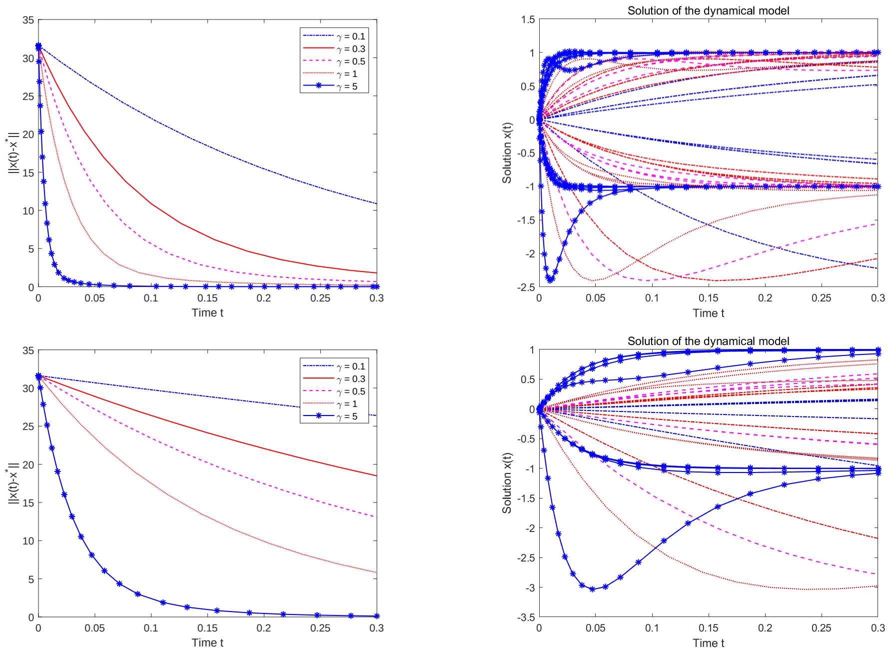

Figure 2: Convergence behaviors of $\|x(t) - x^*\|$ (left figures) and transient behaviors of $x(t)$ (right figures) for Example 4.1 with $n = 1000$ and $x_0 = (0, 0, \cdots, 0)^\top$. The up two subfigures for (3.12) and the others for (3.14).

Table 1: Comparison results for Example 4.1.

n	Model	(3.12)	(3.14)
10000	t_f	2.9	2.3
	CPU	1.6938	0.5044
	Res	5.4372×10^{-7}	9.7293×10^{-7}
15000	t_f	13.8	2.3
	CPU	38.2354	0.7247
	Res	8.7963×10^{-7}	7.5307×10^{-7}
20000	t_f	11.9	2.3
	CPU	37.2243	0.8808
	Res	7.2054×10^{-7}	7.6512×10^{-7}
25000	t_f	13.8	2.3
	CPU	61.0621	1.1111
	Res	7.5087×10^{-7}	8.2130×10^{-7}
30000	t_f	11.3	2.3
	CPU	47.6195	1.2877
	Res	8.8768×10^{-7}	8.3634×10^{-7}

the vectors $x = (x_1, 0)^\top$ with any $x_1 \geq 0$ are equilibrium points of (3.12) and (3.14). According to Theorem 3.5, any solution of (3.12) will converge to an equilibrium point of it. However, the solution of (3.14) may not converge to any equilibrium point of it since A is not positive definite (though it is symmetric). Fig. 3 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^\top$ for (3.12) with 7 different initial points, from which we find that each trajectory generated by the dynamical system (3.12) approaches to a solution of SOCAVEs (1.2). Fig. 4 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^\top$ for (3.14) with 7 different initial points, from which we find that each trajectory generated by the dynamical system (3.14) does not approach to a solution of SOCAVEs (1.2). Numerical results demonstrate our claims.

Specifically, we have the following monotone properties of the solution of (3.12):

(a) If $x_1 \geq |x_2| \geq 0$, then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -2\gamma x_2 \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq 0, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > 0. \end{cases}$$

(b) If $-|x_2| < x_1 < |x_2|$, then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \gamma(|x_2| - x_1) > 0,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\gamma\left(1 + \frac{x_1}{|x_2|}\right)x_2 \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq 0, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > 0. \end{cases}$$

(c) If $x_1 \leq -|x_2| \leq 0$, then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -2\gamma x_1 \geq 0,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 0.$$

For (3.14), $\mathrm{d}x_1/\mathrm{d}t$ is the same with that of (3.12) while $\mathrm{d}x_2/\mathrm{d}t$ is the negative of that of (3.12). The same goes to Example 4.3.

Example 4.3. Consider SOCAVEs (1.2) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Obviously, SOCAVEs (1.2) has a unique solution $x^* = (0,1)^\top$ for this example. Thus, there is a unique equilibrium point for (3.12) and (3.14), respectively. According to Theorem 3.5, any solution of (3.12) will converge to the unique equilibrium point of it. Fig. 5 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^\top$ for (3.12) with 8 different initial points, from which we find that all of the trajectories generated by the dynamical system (3.12) approach to the unique solution of SOCAVEs (1.2). However, similar to Example 4.2, any solution of (3.14) will not converge to the unique equilibrium point of it since A is not positive definite (though it is symmetric). In addition, we have the following monotone properties of the solution of (3.12):

(a) If $x_1 \geq |x_2| \geq 0$, then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\gamma < 0,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\gamma(-1 + 2x_2) \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq \dfrac{1}{2}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > \dfrac{1}{2}. \end{cases}$$

(b) If $-|x_2| < x_1 < |x_2|$, then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \gamma(-1 + |x_2| - x_1) \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} \geq 0, & \text{if} \quad x_1 \geq 1 - |x_2|, \\ \dfrac{\mathrm{d}x_1}{\mathrm{d}t} < 0, & \text{if} \quad x_1 < 1 - |x_2|, \end{cases}$$

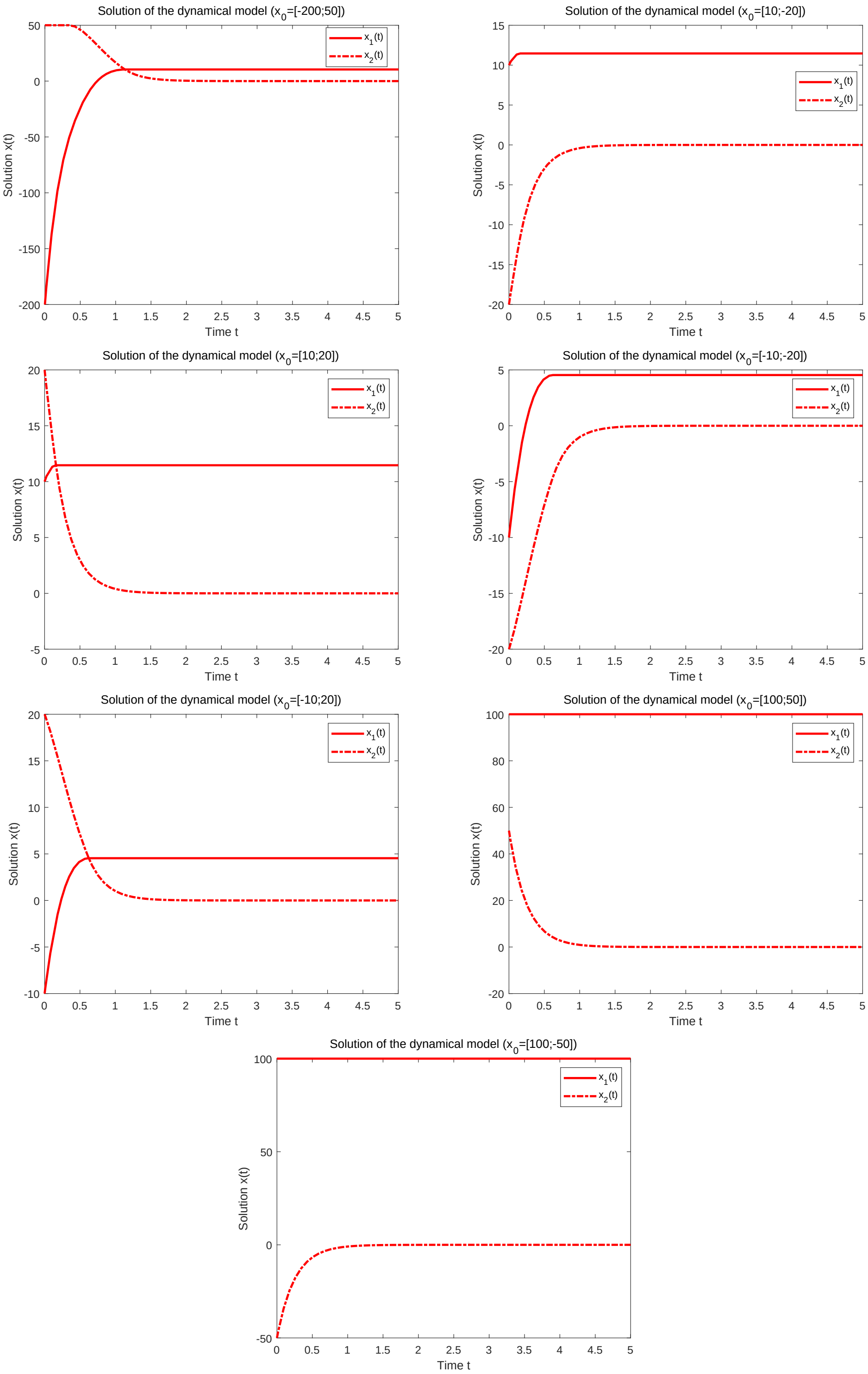

Figure 3: Transient behaviors of (3.12) for Example 4.2 ($tspan = [0, 5]$).

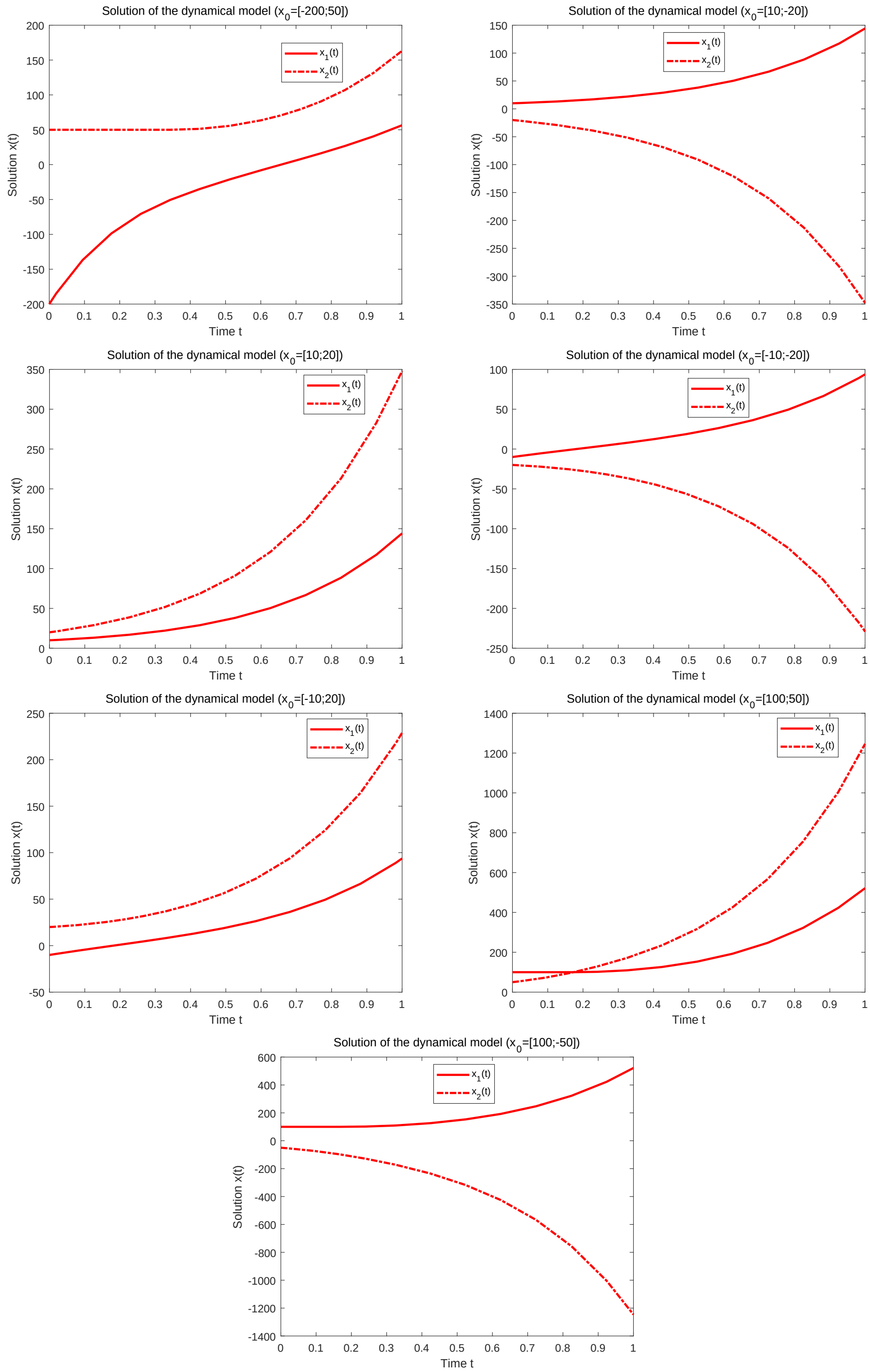

Figure 4: Transient behaviors of (3.14) for Example 4.2 ($tspan = [0, 1]$).

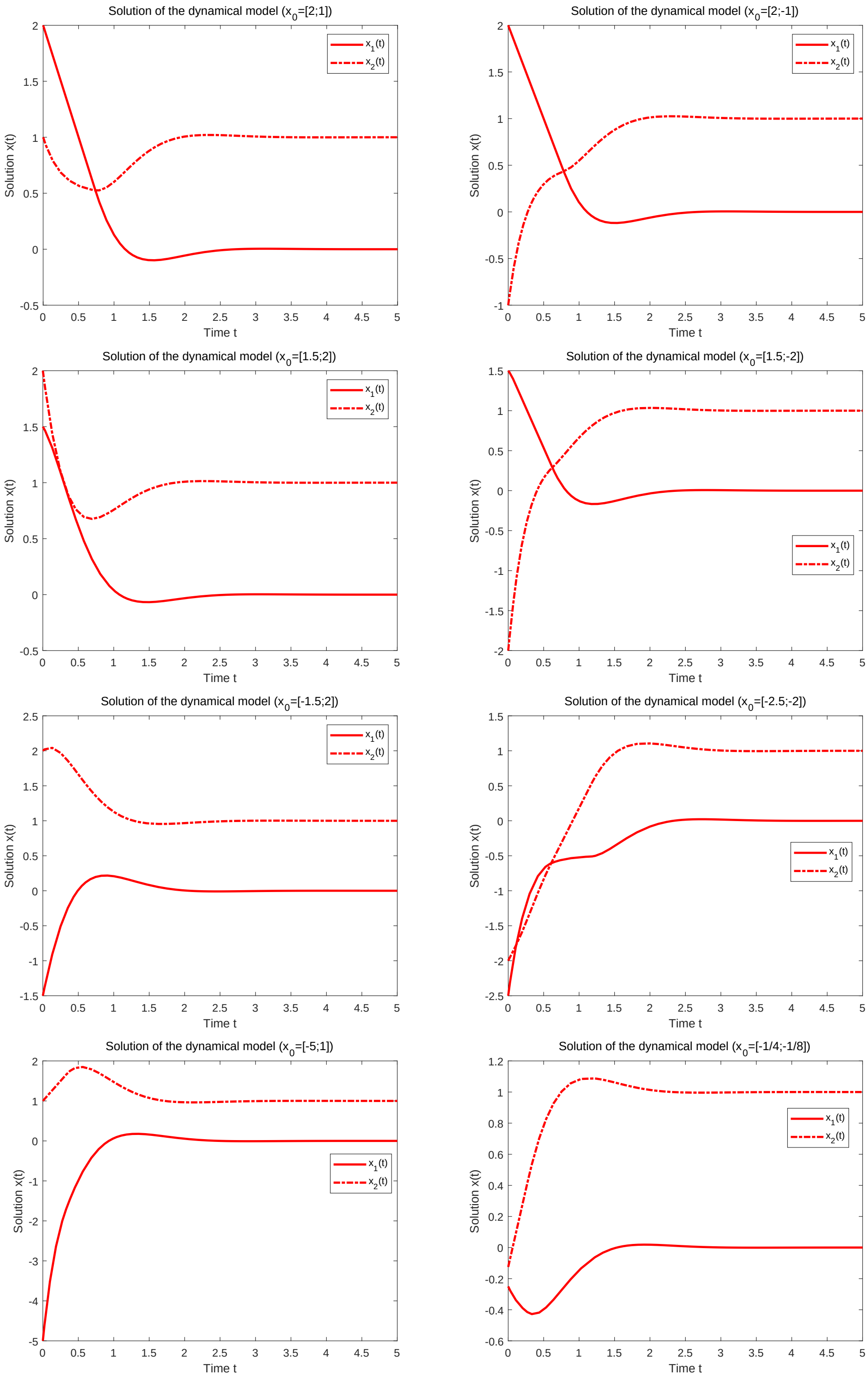

Figure 5: Transient behaviors of (3.12) for Example 4.3 ($tspan = [0, 5]$).

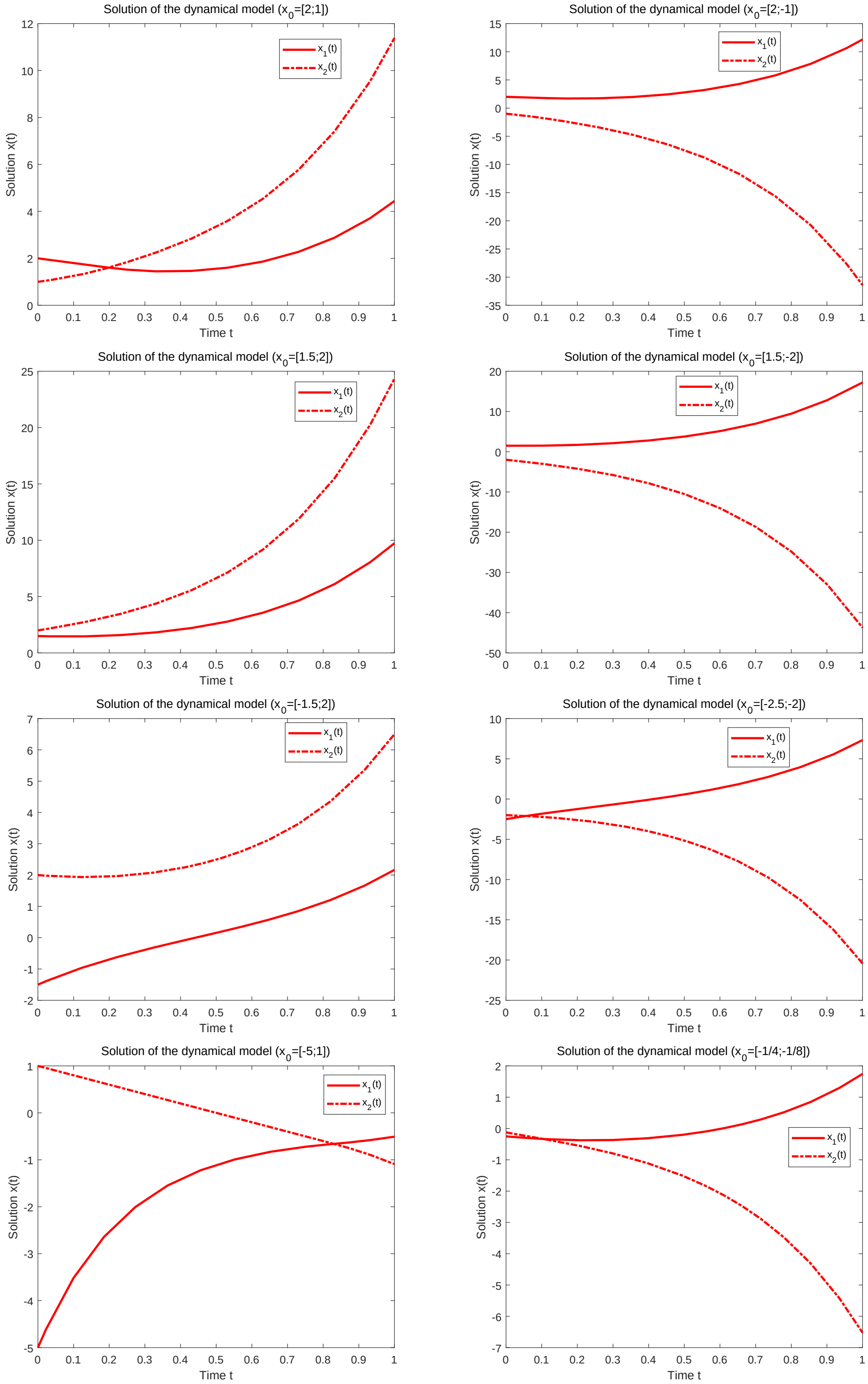

Figure 6: Transient behaviors of (3.14) for Example 4.3 ($tspan = [0, 1]$).

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\gamma\left[-1 + \left(1 + \frac{x_1}{|x_2|}\right)x_2\right] \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq \dfrac{|x_2|}{x_1 + |x_2|}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > \dfrac{|x_2|}{x_1 + |x_2|}. \end{cases}$$

(c) If $x_1 \leq -|x_2| \leq 0$, then

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = \gamma(-1 - 2x_1) \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} \geq 0, & \text{if} \quad x_1 \leq -\dfrac{1}{2}, \\ \dfrac{\mathrm{d}x_1}{\mathrm{d}t} < 0, & \text{if} \quad x_1 > -\dfrac{1}{2}, \end{cases} \\ &\frac{\mathrm{d}x_2}{\mathrm{d}t} = \gamma > 0. \end{aligned}$$

Example 4.4. Consider SOCAVEs (1.2) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

SOCAVEs (1.2) has no solutions for this example. Thus, the dynamical system (3.12) has no equilibrium points. Furthermore, (3.12) is equal to (3.14) for this example since $A^\top = I$. Indeed, we have

(a) If $x_1 \geq |x_2| \geq 0$, then

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \gamma > 0, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \gamma > 0. \end{aligned}$$

(b) If $-|x_2| < x_1 < |x_2|$, then

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = \gamma(1 + |x_2| - x_1) > 0, \\ &\frac{\mathrm{d}x_2}{\mathrm{d}t} = \gamma\left[1 + \left(\frac{x_1}{|x_2|} - 1\right)x_2\right] \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq -\dfrac{|x_2|}{x_1 - |x_2|}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > -\dfrac{|x_2|}{x_1 - |x_2|}. \end{cases} \end{aligned}$$

(c) If $x_1 \leq -|x_2| \leq 0$, then

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = \gamma(1 - 2x_1) > 0, \\ &\frac{\mathrm{d}x_2}{\mathrm{d}t} = \gamma(1 - 2x_2) \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x_2}{\mathrm{d}t} \geq 0, & \text{if} \quad x_2 \leq \dfrac{1}{2}, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} < 0, & \text{if} \quad x_2 > \dfrac{1}{2}. \end{cases} \end{aligned}$$

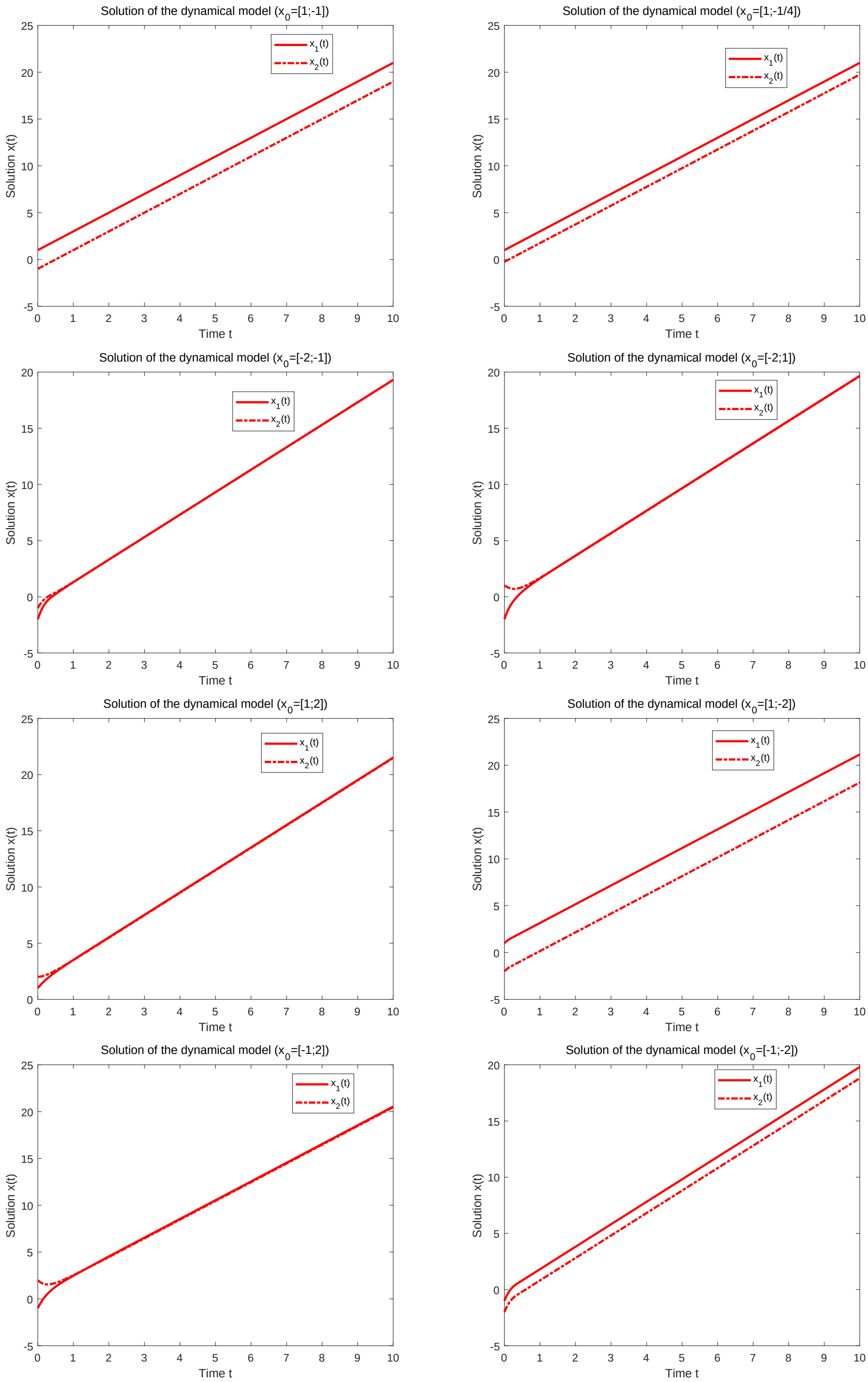

Figure 7: Transient behaviors of (3.12) (and (3.14)) for Example 4.4 ($tspan = [0, 10]$).

Thus, for any initial value x_0, at least $x_1(t)$ in the solution of (3.12) (and (3.14)) is strictly monotonically increasing. Fig. 7 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^\top$ with 8 different initial points, which illustrates our claims.

Examples 4.2 and 4.3 show that the positive definiteness of A is needed for the stability of (3.14). But so far we have not found a solvable SOCAVEs (1.2) with non-symmetric positive definite A and $\|A^{-1}\| \leq 1$ such that the equilibrium point of (3.14) is unstable. We left it as an open question that can we prove the stability of (3.14) under the conditions in Theorem 3.8 without the symmetry of A.

5. Brief conclusion

In this paper, two novel dynamical models are proposed to solve SOCAVEs (1.2), which are different from the existing conventional optimization methods for SOCAVEs (1.2). Theoretical results show that the presented models are globally convergent under certain conditions. Numerical results are given to show the effectiveness of our methods.

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