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# Error Analysis of the Deep Mixed Residual Method for High-order Elliptic Equations

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**Abstract.** This paper presents an a priori error analysis of the Deep Mixed Residual method (MIM) for solving high-order elliptic equations with non-homogeneous boundary conditions, including Dirichlet, Neumann, and Robin conditions. We examine MIM with two types of loss functions, referred to as first-order and second-order least squares systems. By providing boundedness and coercivity analysis, we leverage Céa's Lemma to decompose the total error into the approximation, generalization, and optimization errors. Utilizing the Barron space theory and Rademacher complexity, an a priori error is derived regarding the training samples and network size that are exempt from the curse of dimensionality. Our results reveal that MIM significantly reduces the regularity requirements for activation functions compared to the deep Ritz method, implying the effectiveness of MIM in solving high-order equations.

**AMS subject classifications**: 65N15, 68Q25

**Key words**: Neural network approximation, deep mixed residual method, high-order elliptic equation.

## 1. Introduction

Partial differential equations (PDEs) are of fundamental importance in modeling phenomena across various disciplines in natural science and society. Developing reliable and efficient numerical methods has a long history in scientific computing and engineering applications. Traditional numerical methods, such as finite difference and finite element, have been successfully established and widely applied. However, these methods often encounter challenges when applied to high-dimensional problems, primarily due to high computational costs. In fact, approximating PDE solutions using

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traditional methods incurs a computational cost that grows exponentially with the dimensionality of the problem — a phenomenon commonly referred to as the "curse of dimensionality" (CoD).

In recent years, neural networks have emerged as a promising tool for solving PDEs, demonstrating their potential to address the CoD effectively [3,7,10,18,19,26,31]. Notable approaches include the deep Galerkin method [29] and physics-informed neural networks (PINNs) [26], which employ the PDE residual in a least-squares framework as the loss function. Another notable approach, the deep Ritz method (DRM) [7], leverages the variational form (when available) of the target PDE to define the loss function. More recently, the deep mixed residual method (MIM) [18, 19] has introduced auxiliary networks to approximate the solution derivatives, allowing for exact enforcement of boundary and initial conditions. Compared to DRM and PINN, MIM has shown advantages in certain models, producing better approximations and accelerating the training process. Additionally, MIM offers unique benefits for handling high-order PDEs by transforming complex high-order problems into lower-order representations, thereby reducing computational complexity and improving solution stability.

In this work, we present an error analysis of the MIM for solving high-order elliptic equations using two-layer neural networks. High-order elliptic equations have extensive applications in materials science [5, 12], image processing [1], and elastic mechanics [13]. To illustrate the MIM framework for high-order equations, consider a $2n$-order elliptic equation with general boundary conditions

$$\begin{aligned} &\Delta^n u = f, && x \in \Omega, \\ &B(u, \nabla u, \Delta u, \cdots, \nabla \Delta^{n-1} u) = \boldsymbol{g}, && x \in \partial\Omega. \end{aligned} \tag{1.1}$$

MIM introduces auxiliary networks $\phi_i$ and vector-valued networks $\boldsymbol{\psi}_j$ to approximate $\Delta^i u$ and $\nabla \Delta^j u$ for $0 \le i, j \le n-1$. Combining the squared residual loss with a penalty term yields the mixed residual loss function

$$\begin{aligned} &\|\operatorname{div} \boldsymbol{\psi}_{n-1} - f\|^2_{L^2(\Omega)} + \lambda_1 \|B(\phi_0, \boldsymbol{\psi}_0, \cdots, \boldsymbol{\psi}_{n-1}) - \boldsymbol{g}\|^2_{L^2(\partial\Omega)} \\ &\quad + \lambda_2 \left( \sum_{i=0}^{n-1} \|\nabla \phi_i - \boldsymbol{\psi}_i\|^2_{L^2(\Omega)} + \sum_{i=0}^{n-2} \|\phi_{i+1} - \operatorname{div} \boldsymbol{\psi}_i\|^2_{L^2(\partial\Omega)} \right). \end{aligned} \tag{1.2}$$

This formulation is also referred to as the first-order least squares system and has been used in the finite element method [4]. Moreover, we also introduce the second-order least squares system, where we use networks $\varphi_i$ to approximate $\Delta^i u$. Then, the mixed residual loss function is given by

$$\begin{aligned} &\|\Delta \varphi_{n-1} - f\|^2_{L^2(\Omega)} + \lambda_1 \|B(\varphi_0, \nabla\varphi_0, \cdots, \nabla\varphi_{n-1}) - \boldsymbol{g}\|^2_{L^2(\partial\Omega)} \\ &\quad + \lambda_2 \sum_{i=0}^{n-2} \|\Delta\varphi_i - \varphi_{i+1}\|^2_{L^2(\Omega)}. \end{aligned} \tag{1.3}$$

In this paper, we examine the $2n$-order elliptic equation (1.1) under Dirichlet, Neumann, and Robin boundary conditions. Both first-order and second-order least squares