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A Modularized Algorithmic Framework for Interface Related Optimization Problems Using Characteristic Functions

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Abstract. In this paper, we consider the algorithms and convergence for a general optimization problem, which has a wide range of applications in image segmentation, topology optimization, flow network formulation, and surface reconstruction. In particular, the problem focuses on interface related optimization problems where the interface is implicitly described by characteristic functions of the corresponding domains. Under such representation and discretization, the problem is then formulated into a discretized optimization problem where the objective function is concave with respect to characteristic functions and convex with respect to state variables. We show that under such structure, the iterative scheme based on alternative minimization can converge to a local minimizer. Extensive numerical examples are performed to support the theory.

AMS subject classifications: 65K05, 90C26

Key words: Interface problems, thresholding, characteristic function, convergence analysis.

1. Introduction

Interface related optimization problem is a fundamental problem in many appli-

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cations, including problems in material science [8], image processing [6], topology optimization [34], surface reconstruction [44] and so on. A lot of numerical approaches have been developed to solve such problems, including front track based methods [17, 35], phase-field based methods [8, 12, 25, 33], level set based methods [32], parametric finite element based methods [22], two-stage thresholding [4,5], centroidal Voronoi tessellations (CVT) based methods [10, 11, 26], and many others. Usually, solving such problems includes three ingredients:

1. Representation of the interface (implicit or explicit).

2. Approximation of the objective functional under the representation.

3. Approaches to minimize the approximate objective functional. In particular, the representation of the interface is the most fundamental part of a model or a method for interface related optimization problems.

This paper focuses on a wide class of approximate interface related optimization problems where the interface is implicitly represented by indicator functions of corresponding domains. It is motivated by the MBO method for approximating mean curvature flow using indicator functions [1, 30]. Esedoglu and Otto [15] then develop a novel interpretation using minimizing movement and generalize this type of method to multiphase flow with arbitrary surface tensions. The method has subsequently been extended to deal with many other applications, including image processing [14, 28, 29, 38], problems of anisotropic interface motions [13, 16, 31], the wetting problem on solid surfaces [27, 40–42], and so on.

Recently, based on Esedoglu and Otto's novel interpretation, in [38, 39], the authors develop an efficient iterative convolution thresholding method (ICTM) for image segmentation and extend into topology optimization problems [7, 20] and surface reconstructions from point clouds [36]. In general, the problem could be formulated into

$$\min_{\Theta_i\in\mathcal{S},\Omega_i}\mathcal{E}:=\sum_{i=1}^{n}\int_{\Omega_i}F_i(\Theta_1,\dots,\Theta_n)\,dx+\sum_{i=1}^{n}\lambda_i|\partial\Omega_i|. \tag{1.1}$$

Here, F_i are usually fidelity terms, $\Theta_i=(\Theta_{i,1},\Theta_{i,2},\dots,\Theta_{i,m})$ contains all possible parameters in fidelity terms, $\Omega=\cup_{i=1}^n\Omega_i$, $\mathcal{S}=\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_n\cap\mathcal{S}_1^o\times\mathcal{S}_2^o\times\cdots\times\mathcal{S}_n^o$ as the admissible set of $\Theta=(\Theta_1,\Theta_2,\dots,\Theta_n)$, $\mathcal{S}_i\cap\mathcal{S}_i^o$ as the admissible set of Θ_i where $\mathcal{S}_i^o$ is the admissible set for satisfying some constraints of Θ_i which are dependent on the partition of $\Omega=\cup_{i=1}^n\Omega_i$ and $\mathcal{S}_i$ is the admissible set for satisfying some constraints of Θ_i which are independent on the partition, and λ_i are fixed parameters.

Denote u_i to be indicator functions of Ω_i ($i\in[n]$). As pointed out in [15], when $\tau\ll 1$, the measure of $\partial\Omega_i$ can be approximated by

$$|\partial\Omega_i|\approx\sqrt{\frac{\pi}{\tau}}\sum_{j\in[n],j\neq i}\int_{\Omega}u_iG_\tau*u_j\,dx, \tag{1.2}$$