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On the Recovery of Source Term for Fractional Evolution PDEs by MC-fPINNs

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Abstract. In this paper, we solve the inverse source problem of fractional evolution PDEs by MC-fPINNs. We construct the loss function in terms of the governing equation residual, boundary residual, initial residual and measurement data with noise. Meanwhile, we present a rigorous error analysis of this method. In the experimental section, we present the reconstruction outcomes of the source term for three evolutionary fractional partial differential equations (fPDEs): the evolutionary fractional Laplacian equation, the time-space fractional diffusion equation, and the fractional advection-diffusion equation. These experiments illustrate robust performance of MC-fPINNs in both low-dimensional and high-dimensional scenarios. Our results confirm the effectiveness of MC-fPINNs in solving such inverse source problem, and also provide a theoretical foundation to choose neural networks parameters in this algorithm.

AMS subject classifications: 68T07, 65M12, 62G05

Key words: MC-fPINNs, fractional evolution PDEs, inverse source problem.

1. Introduction

Fractional evolution partial differential equations have emerged as a powerful framework for modeling complex systems where traditional integer-order models fall short. These equations, by incorporating non-local and memory effects via fractional

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derivatives, are particularly suited for describing processes in fields as diverse as acoustic wave propagation with frequency-dependent dissipation [4], viscoelastic constitutive law [20] and porous media [28], just to mention a few examples. In many practical scenarios, the internal source term $f(x,t)$ in fractional evolution PDEs needs to be reconstructed simultaneously with the solution $u(x,t)$ [2, 12]. To solve such an inverse problem of fractional partial differential equations, various classical methods have been proposed in the last few years [5, 27, 33]. However, the nonlocality and singularity of fractional derivatives still necessitate substantial computational cost and memory resources, especially for higher dimensional problems.

More recently, with the great success of deep learning in computer science, several neural network-based solvers have been developed. Among these, physics-informed neural networks (PINNs) [24] have become one of the most applicable methods to solve numerous types of PDEs. By definition, it constructs solution through searching an optimal neural network function to minimize some loss function, which consists of residual term, initial and boundary condition, and fitting error of extra measure data [6, 7]. Based on PINNs, various strategies are further proposed to solve the inverse problem of fPDEs. In [23] Pang *et al.* present the fPINNs method, which utilizes automatic differentiation for the integer-order derivatives of the neural network's output and approximates the fractional derivatives through traditional numerical discretization. However, the dependence on conventional differential techniques increases the computational costs and poses difficulties in high-dimensional cases. Later Yan *et al.* [31] developed the Laplace-fPINNs method. Using the Laplace transform, it first converts the initial time-fractional diffusion equation into a constrained equation in Laplace space, then solves this equation with the original PINNs. After that, an inverse Laplace transform is applied to map the PINNs solution back to the time domain. In their work, on the other hand, only the time-fractional equations have been considered, while a further investigation on the time-space fractional differential equations is still needed.

To avoid classical discretization, Guo *et al.* [10] proposed MC-fPINN to compute fractional derivatives in fPDEs. With the Monte Carlo method, the integral in fractional differentiation for the Caputo fractional derivatives can be well approximated. Based on this, we studied the inverse source problem of the fractional Poisson equation [26], in which a rigorous convergence rate of MC-fPINNs was presented. In this work, we would further extend this method to the inverse source problem for fractional evolution partial differential equations. The introduction of time brings more challenges and necessitates essential adjustments in the formulation, analysis, and numerical treatment of the MC-fPINNs method. Except for the neural network for solution $u(x,t)$, we represent the forcing term $f(x,t)$ as another fully-connected neural network. To optimize these two neural networks, we define a new loss function containing regularization terms for the residuals of the fPDEs and the measurement data at the final time $t = T$. Several numerical examples are shown to demonstrate the effectiveness of MC-fPINNs in solving such evolution equations, including: the time-space fractional diffusion equation, the evolution fractional Laplace equation, and the fractional advection diffusion