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L1 Schemes for Time-Fractional Differential Equations: A Brief Survey and New Development

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Abstract. We summarize the L1 method and its variants for the discretization of the Caputo derivative and develop a weighted L1 (WL1) method. The WL1 method includes the classical L1 method as a special case but improves the L1 method in accuracy while accommodating the intrinsic weak singularity of solutions to fractional differential equations. The WL1 method inherits several properties of the L1 method and the analysis of the new method is presented in a simple way. Compared to several variants of the L1 method, the new method accommodates the intrinsic weak singularity of the time-fractional derivatives in a more flexible way, especially for variable-order fractional differential equation. We also develop the fast WL1 method and apply it to initial and boundary value problems. Numerical simulations and comparisons verify our theoretical analysis and demonstrate the flexibility and efficiency of our method.

AMS subject classifications: 26A33, 65M06, 65M12, 65M15, 35R11

Key words: The weighted interpolation, fractional initial value problem, fractional subdiffusion equation, fast convolution method, variable-order fractional operator.

1. Introduction

In this work, we review the so-called L1 method and its variants for approximating Caputo-fractional derivatives in time and ordinary differential equations and develop a new method based on the L1 method. The Caputo fractional derivative operator ${}_CD_t^{\alpha}$ is defined by [32, p. 79]

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$$
\begin{aligned}
{}_C D_t^{\alpha} u(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\,\mathrm{d}s, \quad 0<\alpha<1, \\
\Gamma(x) &= \int_0^{\infty} t^{x-1} e^{-t}\,\mathrm{d}t.
\end{aligned} \tag{1.1}
$$

In the L1 method for the Caputo fractional derivative, the first-order derivative is approximated by the first-order backward difference method [39]. The convergence order is shown to be $2-\alpha$ in l_∞ norm (defined over mesh grid) when u is smooth enough. Extending L1 method to even high-order methods for smooth solutions can be found in [9, 19, 21, 28], where high-order (piecewise) polynomials are employed to approximate u and its derivative.

However, in typical ordinary differential equations with a Caputo fractional derivative, the solution is not smooth but the solution usually has some intrinsic and identifiable weak singularity structure. For example, consider the following time-fractional initial value problem:

$$
{}_C D_t^{\alpha} u(t) + \lambda u(t) = f(t) \quad \text{for} \quad t \in (0,T] \quad \text{with} \quad u(0)=u^0, \quad \alpha \in (0,1), \tag{1.2}
$$

where $\lambda \geq 0$. The solution of this equation can be represented using the Mittag-Leffler function, see e.g. [32],

$$
u(t) = u^0 E_\alpha(-\lambda t^\alpha) + \int_0^t E_\alpha\big(-\lambda (t-s)^\alpha\big) f(s)\,\mathrm{d}s, \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{1.3}
$$

When $f=0$, the series solution of u contains $t^{k\alpha}$, $k=1,2,\ldots$, which is weakly singular at $t=0$. When f is smooth enough, e.g. in $C^2[0,T]$, we may apply Taylor's formula to f at $t=0$ and we can write the solution in the form of

$$
u(t) = \sum_{k=1}^{\lceil 1/\alpha \rceil -1} c_k t^{\delta_k} + \psi(t) \quad \text{with} \quad \delta_k \in \{i\alpha + j,\ i=1,2,\ldots,\ j=0,1,2,\ldots\}, \tag{1.4}
$$

where $\delta_k < \delta_{k+1}$, $\psi(t) \in C^2[0,T]$, and $c_1,\ldots,c_k \in \mathbb{R}$. A similar representation holds for a nonlinear equation with smooth nonlinear terms [8]. According to the above representations, the L1 method can be modified to accommodate this special weak singularity of solutions and thus to improve the accuracy of time discretization in global error. The modifications include but are not limited to the fitted L1 method [10], the corrected L1 method [44] and the average L1 method [12, 31]. When $u=t^\alpha$, the L1 method on the uniform step size is shown to be of order one in time far from zero, which is obtained for fractional diffusion equation for both smooth and non-smooth initial value [14].

In [4], α-robust error estimates of the L1 method and its variants for solving initial value problems were carefully analyzed. One will find that the present WL1 method is also α-robust, which can be proved using the approach in [4]. Here, we do not pay much attention to the α-robustness of the L1 type methods.