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New Extrapolation Modulus-Based Economical Cascadic Multigrid Method for Variational Inequality Problems with Nonlinear Source Terms

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Abstract. In this paper, we combine a new extrapolation formula with a cascadic multigrid method to solve variational inequality problems with nonlinear source terms. We construct a new extrapolation modulus-based cascadic multigrid method and a new extrapolation modulus-based economical cascadic multigrid method. Based on a new formula for controlling the number of iterations, the convergence is analyzed. Numerical experiments demonstrate the effectiveness of the proposed methods.

AMS subject classifications: 65F10, 65N55, 65N12

Key words: Variational inequality, nonlinear source terms, new extrapolation modulus-based cascadic multigrid method, new extrapolation modulus-based economical cascadic multigrid method, iterative step control formula.

1. Introduction

Variational inequalities and complementary problems have wide applications in various fields, such as traffic networks, economic equilibrium, image processing, and optimization theory, making the study of numerical algorithms highly significant. In practical applications, some free boundary value problems can be formulated as the following variational inequality problem with a nonlinear source term [18]:

$$\begin{aligned} &\text{Find } u \in K(u), \\ &a(u, v-u) + (f(x,y,u), v-u) \geq 0, \quad \forall v \in K(u), \end{aligned} \tag{1.1}$$

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where

$$K(u) = \{u \in H_0^1(\Omega), u \geq 0 \text{ a.e } \Omega\}$$

is a closed convex set in $H_0^1(\Omega)$,

$$a(u,v) = \int_\Omega \nabla u \nabla v dx, \quad f(x,y,u)$$

is continuously differentiable, and $\partial f/\partial u \geq 0$ on $\bar{\Omega} \times \{u : u \geq 0\}$.

Discretizing (1.1) by using the finite element method, the following nonlinear complementarity problem is obtained:

$$u_h \geq 0, \quad Au_h + q + F_h(u_h) \geq 0, \quad u_h^T\big(Au_h + q_h + F(u_h)\big) = 0, \tag{1.2}$$

where $A \in R^{n\times n}$, $q_h \in R^n$, $F(u_h) = (F_1(u_1), \cdots, F_n(u_n))^T$, $\partial F_i(u_i)/\partial u_i \geq 0$, $i = 1, \ldots, n$.

Currently, many valuable research results have been achieved in the numerical methods for solving complementarity problems. Bai [1] proposed the modulus-based matrix splitting iteration method for solving linear complementarity problems, which has proven to be highly effective in practical applications. Xia and Li [24] extended this method to solve nonlinear complementarity problems. Since then, numerous efficient algorithms have been proposed [5, 10, 13–15, 19, 25, 26], etc.

The cascadic multigrid method [2] has garnered attention from researchers due to its simplicity and high computational efficiency, leading to significant achievements, as referenced in [11, 16, 17, 20, 27–29]. In [21], an economical cascadic multigrid method was proposed, reducing the corresponding computational work. Based on the superconvergence theory of finite elements and extrapolation methods, Li *et al.* [12] proposed an efficient cascadic multigrid method. In [3, 4, 8, 9], a new extrapolation cascadic multigrid method was developed based on a new extrapolation formula. The numerical experiments demonstrate the effectiveness of the algorithm.

By combining the advantages of the modulus-based matrix splitting iterative method and the cascadic multigrid method, Wang and Li [23] proposed a modulus-based cascadic multigrid method for solving elliptic variational inequality problems. This method was further extended to solve quasi-variational inequality problems and variational inequality problems with nonlinear source terms in [7] and [6], respectively. Theoretical analysis and numerical results demonstrated its efficiency.

Activated by the ideas from the aforementioned literature, combining new extrapolation formula with the modulus-based cascadic multigrid method, a new extrapolation modulus-based cascadic multigrid method (EXCMG) and a new extrapolation modulus-based economical cascadic multigrid method (EXECMG) for solving variational inequality problems with nonlinear source terms are developed. New methods have two advantages: (1) present the better initial valve for the next finer level by using new extrapolation and high-order interpolation method; (2) get less smoothing steps by new step-choose formula. Numerical experiments demonstrate that the new algorithm exhibits high computational efficiency.