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Variable s-Step Technique for New Conjugate Residual Algorithms for Solving Non-Square Linear Systems Arising in Control Problems

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Abstract. This paper introduces two algorithms based on the conjugate residual method and variable s-step for solving linear systems with non-square coefficient matrices. The introduced algorithms demonstrate their effectiveness through numerical comparisons with other methods, specifically CGNE and CGNR, to solve non-square linear systems. The approach aims to enhance the efficiency of solving problems related to control systems and imaging underground layers, particularly in the context of seismic tomography.

AMS subject classifications: 65F18, 65F10, 65J10

Key words: Non-square linear system, s-step algorithm, control system, least square problem, Krylov subspace method.

1. Introduction

Classical iterative methods for solving linear systems are known for their high computational costs. The Krylov subspace methods (KSMs) has emerged as an alternative approach for classical iterative methods. The conjugate gradient (CG) method, the famous KSM, was initially developed to solve symmetric positive definite linear systems [27]. Other methods, such as SYMMLQ and MINRES, were introduced to find the solution of symmetric (non-positive definite) systems [43]. Subsequently, KSMs were extended to non-symmetric systems, resulting in the development of methods such as

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CGNE, CGNR [5], LSQR [42], GMRES [46], CGS [48], BiCG [35], Bi-CGSTAB [53], among others. A comprehensive presentation of these methods can be found in [40].

In recent years, KSMs have found applications for solving the matrix equations [26], the tensor equations [25], the eigenvalue problems [24] and other related problems. The CG algorithm was utilized for large-scale nonlinear equations and image restoration problems [57]. This algorithm was also employed in the optimal guidance of missile landings [36], and for solving nonlinear problems [2]. In [19], a recursive sequence was proposed to solve parametric systems using the BiCG method with Chebyshev polynomials. The shifted BiCG method was used to solve the Stein matrix equation in [58]. Preconditioned GMRES methods were presented to handle Toeplitz linear systems in fractional eigenvalue problems [61] and ill-posed systems [41]. Several algorithms utilizing Krylov subspace methods and their convergence analyses were developed to solve tensor systems with Einstein products [23, 28].

In 1989, for the first time, Chronopoulos and Gear proposed ts-step CG for solving symmetric linear systems [13]. This method (unlike [54]) allowed stable computation of iterations in s-step CG for $s \leq 5$. The s-step methods require $\mathcal{O}(2s)$ inner products per s-step iteration. The s-step KSMs have several applications in solving practical engineering problems increasing the performance of computations of high performance computing systems (HPCS) [20, 21, 29, 55]. The s-step technique was used for solving linear systems and eigenvalue problems of sparse symmetric and nonsymmetric matrices, also implemented in high-performance computing applications [12, 13, 18, 32, 33, 38]. This technique was also applied to the conjugate residual (CR) method [10]. Then s-step CR method was extended for non-symmetric problems by using modified Gram Schmidt orthogonalization of the direction vectors where is stable for $s \leq 16$ [15–18]. To optimize the runtime of GMRES and orthomin(k) methods, the s-step variant of them was introduced in [11, 14, 17, 30]. In [1, 22, 51, 60], various s-step algorithms were presented to solve several problems.

Recent works in the field of s-step KSMs have introduced novel approaches [9]. In [6], a bound on the difference between the true residual and the updated residual was presented for communication-avoiding Krylov methods, allowing approximation without increasing communication and computation. Subsequently, an s-step Lanczos algorithm was introduced, extending the accuracy band of the Lanczos algorithm to the s-step Lanczos algorithm in [7]. The s-step nonsymmetric Lanczos algorithm was introduced to reduce the synchronization of QMR and BiCG algorithms in [22]. In [8], with a slight increase in computations in the s-step Lanczos algorithm, the accuracy of the CG algorithm is improved. By choosing appropriate polynomials and block orthogonalization, the s-step GMRES algorithm was improved in terms of numerical stability in [56]. For better parallelization of s-step GMRES and s-step orthomin(k) algorithms on supercomputers, these algorithms were introduced with a reduction in communication and appropriate data locality [14].

In [51], the estimation of a suitable fixed value for s in the s-step orthomin(k) algorithm was performed using the KADNA library. However, this fixed value did not enhance the algorithm's flexibility in reducing its communication. Recently, the s-step