Numer. Math. Theor. Meth. Appl. Vol. **18**, No. 3, pp. 619-649
doi: 10.4208/nmtma.OA-2024-0127 August 2025

On the Criteria for Online Updating of Basis Functions for Model Order Reduction Techniques

Han Zhang[1], Zhiyong Wang[1], Zihao Mou[2], Taidong Niu[1] and Helin Gong[3,*]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Sichuan 611731, China*
[2] *School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen 518172, China*
[3] *Paris Elite Institute of Technology, Shanghai Jiao Tong University, Shanghai 200240, China*

Received 1 November 2024; Accepted (in revised version) 11 February 2025

Abstract. The model order reduction and data assimilation techniques have been extensively applied to construct surrogate models as core models for digital twins in different industries. However, in most applications, the online phases rely on a single offline basis construction within the parameter domain, which is computationally expensive and often overlooks local parameter variations due to anisotropic behavior. Consequently, this poses challenges for real-time monitoring where parameters vary dynamically. To address this, we introduce a relative deviation score of reduced basis coefficients, represented by $\mathcal{W}$, to assess the reconstruction capability of basis functions under current parameters. Additionally, we provide a quantitative threshold $\overline{\mathcal{W}}$ for $\mathcal{W}$. Using the reconstruction of a two-dimensional reactor core field distribution as a case study, we demonstrate that if $\mathcal{W}$ remains below the threshold $\overline{\mathcal{W}}=1$, the average reconstruction error can be maintained within the same order of magnitude as the fundamental error limit ε $(\kappa\leq 10)$. Furthermore, we compare $\mathcal{W}$ with the traditional one-time reduced basis selection method and the metric $\mathbf{C}_{R2F}$ [F. Bai and Y. Wang, SN Applied Sciences, 2 (2020), p. 2165], confirming the lower computational cost and enhanced robustness of our proposed algorithm.

AMS subject classifications: 35Q92, 93B07

Key words: POD, model order reduction, data assimilation, parameterized dynamical restriction, 2D LWR problem.

1. Introduction

Digital twins (DTs) have been widely implemented across various industries, includ-

*Corresponding author. *Email addresses:* gonghelin@sjtu.edu.cn (H. Gong), zhywang@uestc.edu.cn (Z. Wang)

ing product design, production, prognostics, and other fields. The core of DTs lies in the seamless integration between the cyber and physical systems using surrogate models, which ensures both speed and accuracy. To meet the speed requirement, reduced-order models (ROM) are often employed, significantly reducing the computational cost of original models such as physics-based finite element models or parameterized partial differential equations. Not only in DTs, ROM has gained popularity in various fields, such as polynomial systems [8] and combustion problems [17]. However, relying solely on ROM may achieve speed at the expense of accuracy, leading to the introduction of data assimilation (DA).

DA, as a critical component of DTs, incorporates measurement information from the physical system. By combining measurements with reduced basis, DTs can update surrogate models in real-time while maintaining high accuracy. In this way, the surrogate models preserve the underlying mathematical structure and enhance system interpretability, while the observational data provide additional insights that may not be captured by the model alone. Originally developed for atmospheric modeling [31], DA has greatly advanced weather forecasting over the past four decades [1,16,40], and more recently has been applied in nuclear engineering for rapid online field reconstruction [3, 24, 28], and is now being integrated with ROM and machine learning (ML) to further enhance predictive capabilities [15, 19, 27, 34]. ML-combined methods represent the recent development of ROM following the traditional intrusive ROM (IROM) and non-intrusive ROM (NIROM), referred to as the physics and data-driven dual ROM. By incorporating machine learning techniques, such as long short term memory (LSTM) and physics-data combined neural network (PDCNN), to assist in driving the ROM, such approach has been demonstrated to possess stronger extrapolation and generalization capabilities, especially for nonlinear parametric PDEs [18, 39].

A common framework for DA with reduced basis and noisy measurements typically consists of two key stages. The first stage, referred to as the offline phase, focuses on extracting basis functions from a set of snapshots. These snapshots, known as the solution manifold, are generated by solving the system using representative parameters. The system state is then approximated as a linear combination of n reduced modes, forming what is known as the reduced-order model. Various approaches have been developed for this purpose. For instance, Buffa *et al.* [14] introduced the greedy algorithm, and Barrault *et al.* [7] proposed the empirical interpolation method (EIM). Additionally, matrix QR factorization, as discussed by Trefethen and Bau [44], is frequently used for constructing the solution manifold and deriving the reduced basis functions. The second stage, referred to as the online phase, focuses on the estimation of coefficients for the n selected modes using observational data. In this stage, the system's state is approximated by a linear combination of the reduced basis functions obtained from the offline phase. When the number of observation vectors, m, equals the number of modes, $m = n$, interpolation is employed. However, for $m > n$, least-squares regression is applied to determine the coefficients. This is a key step in reduced-order modeling, where the goal is to minimize the difference between the predicted and observed data while maintaining computational efficiency. Binev *et al.* [9]