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Implicit-Explicit Time Discretization Schemes for a Class of Semilinear Wave Equations with Nonautonomous Dampings

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Abstract. This paper is concerned about the implicit-explicit (IMEX) methods for a class of dissipative wave systems with time-varying velocity feedbacks and nonlinear potential energies, equipped with different boundary conditions. Firstly, we approximate the problems by using the Hochbruck-Leibold IMEX method, which is a second-order scheme for the problems when the damping terms are time-independent. However, rigors analysis shows that the error rate declines from second to first order due to the nonautonomous dampings. To recover the convergence order, we propose a modified IMEX scheme and apply it to the nonautonomous wave equations with a kinetic boundary condition. Our numerical experiments demonstrate that the modified scheme can not only achieve second-order accuracy but also improve the computational efficiency.

AMS subject classifications: 65M12, 65M15, 65J08

Key words: Semilinear wave equations, nonautonomous damping, dynamic boundary condition, IMEX, error analysis.

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1. Introduction

In this paper we consider numerical solutions for the following semilinear evolution equation of second-order:

$$\ddot{u}(t) + B(t)\dot{u}(t) + Au(t) = f\big(t, u(t)\big)$$

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in a suitable Hilbert space. Such evolution equation is renowned for their prevalence in a spectrum of mathematical models derived from physics, elastodynamics, acoustics etc, for example, the propagation of dislocations in crystals [24], wave motions in a dissipative medium [5], and structural acoustic interactions [20], thereby providing a versatile abstract framework that encompasses a multitude of wave-type equations wave-type equations with a diverse array of boundary conditions such as Dirichlet, Neumann, Robin and dynamic boundary conditions.

The theoretical exploration of the semilinear evolution equation has attracted many researchers' attention over an extended period, while the numerical analysis on this evolution equation can be broadly classified into two main categories based on the boundary conditions (cf. [4, 22, 28, 29, 31, 32, 36]). In the first category, it is concerned with wave-type equations subject to Dirichlet, Neumann or Robin boundary conditions. There are numerous investigations on numerical methods for these equations including finite difference methods [6,8], finite element methods [14,21], characteristic methods [3, 13], spectral methods [9, 35] and the methods of lines [33, 34], among others. The second category focuses on wave equations with dynamic boundary conditions. A series of papers have been published recently to provide a comprehensive error analysis on some discrete schemes for this type of wave equations [15–17,26]. It is noteworthy that the damping terms are time-independent in all these papers. In light of this, this paper is aimed to investigate the numerical schemes for semilinear wave-type equations with time-varying dampings and subject to dynamic boundary conditions, addressing a gap in the current literature.

It is a well-established fact that explicit schemes usually suffer severe time step restrictions despite their computational simplicity, and full implicit schemes are always unconditionally stable but rather taxing and time consuming. Against this backdrop, the focus of this paper is on an IMEX time discretization scheme. This approach is a popular and widely-used approach (cf. [1, 11, 23, 27]), which leverages an implicit treatment for the unbounded linear components and an explicit treatment for the nonlinear aspects of a differential equation. For an in-depth investigation of the IMEX scheme, we direct the readers to seminal works: for IMEX Runge-Kutta schemes [2], for IMEX multistep schemes [10], for Crank-Nicolson-leapfrog IMEX scheme [25], and references therein.

Closely pertinent to our investigation, we refer to a recent study [18], where an IMEX scheme is introduced. This scheme is crafted by merging the Crank-Nicolson method for linear segments and the leapfrog method for nonlinear components, culminating in a second-order numerical scheme. However, when this IMEX scheme is applied to a class of semilinear wave equations with nonautonomous damping terms, it is noted that the error rate descends to the first order, as will be elaborated in Section 4. Thus, it is both natural and necessary to delve into the influence that nonautonomous damping terms exert on the convergence rate of the IMEX scheme.

Motivated by previous research, this paper aims to present a comprehensive discretization of the aforementioned semilinear evolution equation in an abstract manner. The major contributions of this article can be summarized as follows: