Numer. Math. Theor. Meth. Appl. Vol. **18**, No. 3, pp. 680-702
doi: 10.4208/nmtma.OA-2024-0139 August 2025

Solving Generalized Tensor Eigenvalue Problem via Spectral Shifted Inverse Power Methods

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Received 15 December 2024; Accepted (in revised version) 17 February 2025

Abstract. In this paper, we propose a generalized eigenproblem algorithm, called spectral shifted inverse power method (SIPM) for computing tensor generalized eigenvalue. The SIPM method is developed to overcome the limitations of existing approaches, such as the generalized eigenproblem adaptive power, tensor Noda iteration, modified tensor Noda iteration, and generalized Newton-Noda iteration. These methods often suffer from slow convergence, sensitivity to initial conditions, and computational inefficiency. First, we express SIPM as a fixed point iteration form and establish the connection between the fixed points and generalized eigenvectors of symmetric tensors. Moreover, we introduce a shift power method that further enhances SIPM. Next, we provide a technique for selecting the optimal starting point. Finally, we present numerical results that confirm the effectiveness of our method in solving the tensor generalized eigenvalue problem more efficiently than other methods.

AMS subject classifications: 15A18, 15A69

Key words: Spectral shifted inverse power method, shifted symmetric higher-order power method, Z-eigenpair, H-eigenpair, generalized tensor eigenvalue problem.

1. Introduction

Tensors, which are multidimensional arrays, have gained significant importance in various fields of applied and computational mathematics [2,3,34]. They play a crucial role in advancing numerical multilinear algebra and exhibit a wide range of practical applications, particularly in chaotic image encryption, image super-resolution, and the Volterra model [20]. Among the family of tensors, third-order tensors are particularly dominant in applications [7,13,17]. A notable example of their successful application

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is found in the study of liquid and solid crystals. The authors in [6] introduced two significant concepts: C-eigenvalues and C-eigenvectors.

Suppose $\mathcal{A} \in \mathbb{R}^{[3,n]}$ be a piezoelectric-type tensor, and let $x, y \in \mathbb{R}^n$ be two unit-length vectors, i.e.,

$$x^T x = 1, \quad y^T y = 1.$$

If there exists a real scalar λ satisfying the equations $Ayy = \lambda x$ and $xAy = \lambda y$, then λ is defined as a C-eigenvalue of the piezoelectric-type tensor $\mathcal{A}$. Additionally, x and y are referred to as the left and right C-eigenvectors corresponding to λ.

Liu and Mo [20] defined a symmetric tensor $\mathcal{B} = (b_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,n]}$, with entries given by

$$b_{i_1 i_2 i_3 i_4} = \sum_{i \in [n]} a_{i_1 i_2} a_{i i_3 i_4}.$$

They established a connection between the C-eigenpairs of a piezoelectric-type tensor and the Z-eigenpairs of the constructed fourth-order symmetric tensor. By computing the non-negative Z-eigenvalues of this constructed tensor, they determined all the C-eigenvalues of the original tensor and obtained the corresponding C-eigenvectors. However, since the introduction of the generalized tensor eigenvalue problem by Chang *et al.* [4], relatively little research has focused on generalized eigenproblems. Two numerical algorithms for the generalized tensor eigenproblem were proposed by Kolda and Mayo in 2014 [19]. They employed an adaptive shifted power method to solve the optimization problem

$$\max_{\|x\|=1} \frac{\mathcal{A}x^m}{\mathcal{B}x^m} \|x\|^m.$$

This approach extends the shifted symmetric higher-order power method (SS-HOPM) for finding Z-eigenpairs by introducing an adaptive method for automatic shift selection.

In 2016, Yu *et al.* [30] introduced the adaptive gradient (AG) method for computing tensor generalized eigenvalues, demonstrating linear convergence under specific conditions. Zhao *et al.* [33] later proposed two convergent gradient projection methods for solving the generalized eigenvalue problem of weakly symmetric tensors. Among them, the gradient projection (AGP) method is comparable to a modified the generalized eigenproblem adaptive power (GEAP) approach, while the spectral gradient projection method outperforms GEAP, AG, and AGP by incorporating the Barzilai-Borwein (BB) method with the gradient projection strategy.

Furthermore, Chen *et al.* [5] proposed a linear homotopy method for solving the generalized eigenproblem and proved that it finds all isolated eigenpairs. In recent years, several optimization approaches have been developed for tensor eigenvalue problems [11, 12, 14, 15, 23, 24, 31]. Han introduced an unconstrained optimization model [11] for computing generalized eigenpairs of symmetric tensors, while a subspace projection method was proposed in [12] specifically for Z-eigenvalues.

The generalized eigenvalue (GE) problem is of great importance in various fields, including science, engineering, and machine learning. In 2021, Liang *et al.* [26] intro-