

A Modified Version of the PRESB Preconditioner for a Class of Non-Hermitian Complex Systems of Linear Equations

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Abstract. We present a modified version of the PRESB preconditioner for two-by-two block systems of linear equations with the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} F & -G^* \\ G & F \end{pmatrix},$$

where $F \in \mathbb{C}^{n \times n}$ is Hermitian positive definite and $G \in \mathbb{C}^{n \times n}$ is positive semidefinite. Spectral analysis of the preconditioned matrix is analyzed. In each iteration of a Krylov subspace method, like GMRES, for solving the preconditioned system in conjunction with proposed preconditioner two subsystems with Hermitian positive definite coefficient matrix should be solved which can be accomplished exactly using the Cholesky factorization or inexactly utilizing the conjugate gradient method. Application of the proposed preconditioner to the systems arising from finite element discretization of PDE-constrained optimization problems is presented. Numerical results are given to demonstrate the efficiency of the preconditioner. Our theoretical and numerical results show that the proposed preconditioner is efficient when the norm of the skew-Hermitian part of G is small.

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1. Introduction

We are concerned with the following two-by-two block system of linear equations [6, 35, 36]:

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$$\mathbf{Ax} = \begin{pmatrix} F & -G^* \\ G & F \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{b}, \quad (1.1)$$

where $F \in \mathbb{C}^{n \times n}$ is Hermitian positive definite (HPD), $G \in \mathbb{C}^{n \times n}$ is positive semidefinite (PSD) (in general, non-Hermitian), and $p, q, x, y \in \mathbb{C}^n$ in which x, y are unknowns to be computed. We recall that the matrix $A \in \mathbb{C}^{n \times n}$ is said to be positive definite (PD) if $\Re(x^*Ax) > 0$, for every nonzero $x \in \mathbb{C}^n$ (see [14]). Here, the real part of a complex number z is denoted by $\Re(z)$. Similarly, a matrix $A \in \mathbb{C}^{n \times n}$ is said to be positive semidefinite if $\Re(x^*Ax) \geq 0$, for every $x \in \mathbb{C}^n$. It is straightforward to prove that the matrix $A \in \mathbb{C}^{n \times n}$ is PD (resp. PSD) if and only if the matrix $A + A^*$ is HPD (resp. Hermitian positive semidefinite). Naturally, a positive definite matrix which is also Hermitian, is called Hermitian positive definite. Similarly, a Hermitian positive semidefinite (HPSD) matrix is defined.

We assume that the matrix $\mathbf{A}$ is large and sparse. So using direct methods such as Gaussian elimination to solve the above system can be computationally expensive due to the large number of operations required. Therefore, for large and sparse systems of linear equations, iterative methods are often the method of choice for obtaining a solution efficiently.

For two matrices $F, G \in \mathbb{R}^{n \times n}$, and vectors $x, y, p, q \in \mathbb{R}^n$, the complex system

$$(F + iG)(x + iy) = p + iq, \quad (1.2)$$

where $i = \sqrt{-1}$ is the imaginary unit, can be equivalently written in the real form as

$$\begin{pmatrix} F & -G \\ G & F \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}. \quad (1.3)$$

As we see, in this case, if the matrix G is symmetric positive semidefinite and F is symmetric positive definite, then the system (1.1) is a real equivalent form of system (1.2). Equations of the form (1.2) are commonly encountered in scientific computing and engineering applications. For instance, they arise in the numerical solution of the Helmholtz equation and time-dependent partial differential equations (PDEs) [17], diffuse optical tomography [1], algebraic eigenvalue problems [26], molecular scattering [27], structural dynamics [20], and lattice quantum chromodynamics [21].

In the case when F and G are respectively symmetric positive definite (SPD) and symmetric positive semidefinite (SPSD), there are several methods for solving Eqs. (1.2) and (1.3). For example, based on the Hermitian/skew-Hermitian splitting (HSS) method [13], Bai *et al.* [11] presented a modified version of the HSS iterative method, called MHSS, to solve systems of the form (1.2). Next, Bai *et al.* [12] established a preconditioned version of the MHSS method for solving the system (1.3). Salkuyeh *et al.* [32] solved the system (1.3) by the generalized successive overrelaxation (GSOR) iterative method and then Hezari *et al.* [22] proposed a preconditioned version of the GSOR method. The scale-splitting (SCSP) iteration method for solving (1.2) was presented by Hezari *et al.* [23]. Using the idea of the SCSP iteration method, Salkuyeh [30] set up a two-step scale-splitting (TSCSP) for solving Eq. (1.2) and then Salkuyeh and