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Approximate Method for Computing Hypersingular Integrals with Oscillatory Kernels

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Abstract. An extrapolation method is proposed for the numerical computation of hypersingular integrals with oscillatory kernels. The oscillatory integral is reformulated as the weighted integral of a Hadamard finite part, which is subsequently approximated using the weighted trapezoidal rule. The asymptotic expansion of the error function is derived, and both the convergence order and the posterior error of the algorithm are analyzed. Numerical examples verify the theoretical results and demonstrate the validity of the proposed method.

AMS subject classifications: 65R10, 65D30, 65D32, 42A50, 42B20, 65B05

Key words: Oscillatory kernel, composite rectangle rule, hypersingular integral, extrapolation method.

1. Introduction

Highly oscillatory problems are common in many scientific and engineering computations, especially in the study of electromagnetic and acoustic wave scattering. These problems often involve integrals with rapidly oscillating kernels, which are also prevalent in quantum mechanics [2, 14]. Moreover, these problems often involve singularities, where the kernel function exhibits a singular behavior at certain points, such as $1/(x-s)^m$. The integral kernel functions often have not only rapid oscillations but also singularities [17, 30]. These characteristics make the numerical solution of such integrals both challenging and crucial.

In solving partial differential equations (PDEs), the boundary element method (BEM) is often used to reduce a two-dimensional problem to a one-dimensional Fred-

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holm integral equation [21], given by

$$\lambda u(s) + \int_a^b \frac{K(s,x)}{(x-s)^m} u(x)\,dx = f(s), \quad s \in (a,b), \quad m = 1,2,\ldots.$$

Many studies have focused on the numerical solution when $K(s,x) = 1$. However, in practical applications, kernel functions are often highly oscillatory, such as $K(s,x) = e^{ik(x-s)}$, with $k \gg 1$, which leads to the emergence of highly oscillatory integrals. Specifically, oscillatory integrals can be expressed as

$$I(f,s,k) = \int_{-1}^{1} \frac{f(x)\omega(x)}{(x-s)^{m+1}} e^{ikx}\,dx, \quad s \in (-1,1).$$

When $m = 0$, the integral is a Cauchy principal value integral, while for $m \geq 1$ and $m \in N^+$, it becomes a hypersingular integral [19, 20, 22]. These oscillatory integrals pose significant numerical challenges, especially when k is very large, as traditional numerical integration methods often fail to handle them efficiently. Furthermore, some equations have solutions that do not include singular kernels like $1/(x-s)^m$, but instead take the form of more general oscillatory integrals, such as

$$I = \int_a^b f(x) e^{i\omega g(x)}\,dx.$$

Many special functions can be expressed as integrals of highly oscillatory functions, such as Bessel and Hankel functions, sine and cosine integrals, exponential integrals, and hypergeometric functions. The oscillatory kernel is not limited to weighted integrals with exponential functions, but represents a broader class of oscillatory situations that arise in various fields. The model considered in this paper is the general form of highly oscillatory integrals.

In many situations, the study of high-frequency problems essentially translates to analyzing highly oscillatory differential or integral equations, which often involve extensive computations of oscillatory integrals. To address these challenges, a variety of methods have been developed for efficiently computing highly oscillatory integrals. Some of the most widely used techniques include the Filon method [10], the Levin method [16, 25], Clenshaw-Curtis quadrature rules [11, 21], modulated Fourier expansions [4], and Gaussian correlation quadrature formulas [13].

Sloan [24] introduced a stable Clenshaw-Curtis-Filon method based on Chebyshev polynomial approximations. The main drawbacks of the Levin-type methods are that the integration interval cannot include zeros and the computational complexity is relatively high. Evans [9] proposed a generalized integration rule for computing highly oscillatory integrals. In applications involving high-frequency power sources, circuit systems often exhibit highly oscillatory solutions. Condon [5] addressed these problems by providing numerical solutions for simple circuits using a Filon-type method based on generalized Fourier transforms. The combination of boundary element methods with the fast multipole method (FMM) and the Clenshaw-Curtis-Filon method has